Sample Questions

1. State and prove the classical Peano existence theorem for

\[ y' = f(t, y), \quad y(t_0) = y_0, \]

where \( f \) is continuous.

2. State and prove the classical Cauchy existence and uniqueness theorem for

\[ y' = f(t, y), \quad y(t_0) = y_0, \]

where \( f \) and \( \frac{\partial f}{\partial y} \) are continuous.

3. Verify that the system

\[
\begin{align*}
x' &= \cos(xy) - x, \\
y' &= -y + x^2 + 1
\end{align*}
\]

has an equilibrium point at

\[ (x, y) \approx (0.632639, 1.4003) \]

and write down the linearization of the system around this equilibrium solution. Analyze the behavior of the linearization and relate it to the behavior of non-linear system near the equilibrium point.

4. Consider the Runge-Kutta numerical scheme

\[
\begin{align*}
k_1 &= hf(x_n, y_n), \\
k_2 &= hf(x_n + h/2, y_n + k_1/2), \\
k_3 &= hf(x_n + h/2, y_n + k_2/2), \\
k_4 &= hf(x_n + h, y_n + k_3), \\
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{align*}
\]
(a) Compute the local truncation error.

(b) Count the number of function evaluations required for a single step of this scheme.

(c) Make appropriate assumptions on the function $f$ and estimate the global error involved in using this scheme to solve the IVP

$$ y' = f(t, y), \; y(0) = y_0 $$

on the interval $[0, L]$.

(d) Perform a stability analysis of the scheme.

5. Consider the linear multistep method

$$ y_{n+1} = y_n + \frac{h}{12} (23y'_n - 16y'_{n-1} + 5y'_{n-2}). $$

(a) Compute the local truncation error.

(b) Count the number of function evaluations required for a single step of this scheme.

(c) Make appropriate assumptions on the function $f$ and estimate the global error involved in using this scheme to solve the IVP

$$ y' = f(t, y), \; y(t_0) = y_0 $$

on the interval $[0, L]$.

6. Give an example of a stiff system of ODE. Explain why a general numerical scheme is unlikely to provide an accurate solution to a stiff system.