Sample questions

1.
   (a) Complete the following definition:

   Let *f* be a real-valued function defined on a set $S \subset \mathbb{R}$. *f* is <u>uniformly continuous on *S*</u> if ...

   (b) Prove that $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty]$ satisfies the definition of uniform continuity.

   (c) Give an example of a function which is continuous, but not uniformly continuous. Prove that your function does not satisfy the definition of uniform continuity.

2.
   (a) Prove that if $\sum a_n$ is a convergent series with $a_n \geq 0$ and $p \geq 1$, then $\sum |a_n|^p$ converges.

   (b) Is the theorem true if we remove the restriction that $a_n \geq 0$? If true, prove it. If not, give a counterexample.

3.
   (a) Prove that if $\lim s_n = a$ and $\lim t_n = b$ ($a, b < \infty$), then $\lim s_n t_n = ab$.

   (b) Prove that if $\lim s_n = \infty$ and $\liminf t_n > 0$, then $\lim s_n t_n = \infty$.

   (c) Prove that if $\limsup s_n = \infty$ and $\liminf t_n > 0$, then $\limsup s_n t_n = \infty$.

4.
   (a) State the Mean Value Theorem.

   (b) Determine whether the Mean Value Theorem holds for the following functions on the specified intervals. If the conclusion holds, give an example of an interior point which satisfies the theorem. If the conclusion fails, state which hypothesis of the Mean Value Theorem fails.

   i. x^2 on [-1,2]

   ii. $\sin(x)$ on $[0, \pi]$

   iii. $|x|$ on [-1,2]

   iv. $\frac{1}{x}$ on [-1,1]

   v.

$\frac{1}{x}$ on [1,3]

vi.

$\mathrm{sgn}(x)$ on [-2,3], where $\mathrm{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$

(c)

Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbf{R}$.

5.

(a)

State Taylor's remainder theorem.

(b)

Use Taylor's remainder theorem to prove that the Taylor series for $f(x) = \ln(1 + x)$ converges at x=1.

6.

(a)

Prove the Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

(b)

Consider the sequence (x_n) defined recursively as follows:

$$\begin{aligned} x_1 &= 1, \\ x_n &= (2x_{n-1} + 3)/4, \qquad n \geq 2. \end{aligned}$$

Show that (x_n) converges and find the limit.

(c)

Investigate the convergence of (x_n), given $x_n = \sqrt{n+1} - \sqrt{n}$

7.

(a)

State the definition of a Cauchy sequence and show that every convergent sequence is a Cauchy sequence.

(b)

Show directly (from the definition) that if

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

then (x_n) is not a Cauchy sequence.

(c)

Show directly (from the definition) that if

$$x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!},$$

then (x_n) is a Cauchy sequence. (Hint: first show that $n! < 2^{n-1}$ $\forall n$.)

8.

(a) Prove that ``ϵ-δ continuity'' implies sequential continuity, i.e.,

Let $a \in \mathrm{Dom} f$. Suppose that for any $\epsilon > 0$ there exists a δ such that if $y \in \mathrm{Dom} f$ satisfies $|y-a| < \delta$, then $|f(y)-f(a)| < \epsilon$. Prove that for any sequence $(x_n) \in \mathrm{Dom} f$ with $\lim x_n = a$, we have $\lim f(x_n) = f(a)$.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x, & x \text{ irrational} \\ 1-x, & x \text{ rational.} \end{cases}$$

Prove that *f* is continuous at $x=1/2$ and discontinuous everywhere else.

9.

(a) Define: ``uniform convergence'' of a sequence of functions (f_n) defined on a set *D*.

(b) Prove that if f_n is continuous on *D* $\forall n$ and $\lim f_n = f$ uniformly on *D*, then *f* is continuous on *D*.

(c) Give an example of a sequence of continuous functions f_n on a set *D* such that the pointwise limit $f(x) = \lim f_n(x)$ is defined on *D*, but *f* is NOT continuous on *D*.

10.

(a) State the definition of the Riemann integral of a bounded function *f* over an interval [*a*,*b*].

(b) Prove that any continuous function *f* is Riemann integrable on [*a*,*b*]. (Your proof should use the notion of uniform continuity.)

OR

(c) Prove that if *f* is monotone increasing on [*a*,*b*], then $\int_a^b f(x)dx$ exists.

(d) Show that

$$\lim_{n\to\infty} n\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\frac{1}{(n+n)^2}\right\}=\frac{1}{2}$$

by interpreting the sums as Riemann sums for the definite integral of some continuous function over [0,1].

11.

(a)

Let $f(x)=\begin{cases} x^2 & \text{if} \quad x\geq 0 \\ ax & \text{if} \quad x<0. \end{cases}$

i. For what values of *a* is *f* differentiable at *x*=0?

ii. For what values of *a* is *f* continuous at *x*=0?

iii. When *f* is differentiable at *x*=0, does *f*''(0) exist?

(b)

Let $f:\mathrm{R}\to\mathrm{R}$ be defined by the property

$$f(x+y)=f(x)\cdot f(y) \qquad \forall x,y\in are.$$

Suppose that *f* is continuous at zero. Show that *f* must be continuous everywhere.

12.

(a)

Find the radius of convergence of each power series:

i.

$$\sum_0^\infty \frac{(n!)^2}{(2n)!}z^n$$

ii.

$$\sum_0^\infty (2-(-1)^n)^n z^n$$

iii.

$$\sum_0^\infty \frac{n^n}{n!}z^n$$

(b)

Show that $\sum_0^\infty (n+1)^2 x^n = \frac{1+x}{(1-x)^3}$ and state the region of validity. (Hint: Start with the Maclaurin series for $f(x)=\frac{1}{1-x}$.)

13.

(a)

Examine each series for convergence/divergence:

i.

$$\sum_{2}^{\infty} \frac{1}{\ln k}$$

ii.

$$\sum_{2}^{\infty} \frac{1}{k(\ln k)^2}$$

(b)

Prove that the series

$$\left(\frac{1}{2}\right)^p + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^p + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^p + \cdots$$

converges if $p>2$ and diverges if $p \leq 2$.