Sample Problems

1. For any $n \times n$ matrix *A*, the trace of *A* is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

   (a) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $n \times n$ matrices *A* and *B*.

   (b) Prove that, for all $n \times n$ matrices *A* and *B* and all scalars α and β,

$$\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B).$$

   (c) Prove that an $n \times n$ matrix *A* satisfies $\operatorname{tr}(AA^T) = 0$ if and only if *A*=0.

2. Let *A* be an $m \times n$ matrix. Prove that

$$\operatorname{Null}(A)^{\perp} = \operatorname{Col}(A^T).$$

   Here Null(*A*) denotes the *null space* (*kernel*) of *A*, while Col(A^T) is the *column space* (*range*) of A^T.

3. The 3×4 matrix *A* is given by

$$A = 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} 4 & 1 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} -5 & 4 & 2 & 2 \end{pmatrix}.$$

   (a) Find the Singular Value Decomposition (SVD) of *A*.

   (b) Find $A^{\dagger}$, the Moore-Penrose generalized inverse of *A*(you can express it in factored form, if convenient).

(c)

Find the least-squares solution of $Ax=b$, where

$$b = \begin{bmatrix} 20 \\ 1 \\ 52 \end{bmatrix}.$$

4.

Let V be the vector space of real polynomials of degree less than or equal to 2. Define an inner product $<\cdot,\cdot>$ on V by

$$<p,q>= \int_0^1 p(x)q(x)\,dx.$$

(a)

Use the Gram-Schmidt (or modified Gram-Schmidt) procedure to produce an orthogonal basis for V from the standard basis $\{1, x, x^2\}$.

(b)

Find the coordinates of $p(x)=6x^2$ in the orthogonal basis you just computed.

5.

The set $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$, where

$$a_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ -5 \end{bmatrix}, \; a_2 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \; a_3 = \begin{bmatrix} 4 \\ -5 \\ 2 \\ 2 \end{bmatrix}, \; a_4 = \begin{bmatrix} 4 \\ 2 \\ -5 \\ 2 \end{bmatrix},$$

is an orthogonal basis for $\mathbf{R}^4$, while $\mathcal{B} = \{b_1, b_2, b_3\}$, where

$$b_1 = \begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix}, \; b_2 = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}, \; b_3 = \begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix},$$

is an orthogonal basis for $\mathbf{R}^3$. The 3×4 matrix A is defined by

$$A=2\begin{bmatrix}4\\-1\\8\end{bmatrix}\begin{bmatrix}4 & 2 & 2 & -5\end{bmatrix}+\begin{bmatrix}-7\\4\\4\end{bmatrix}\begin{bmatrix}1 & 4 & 4 & 4\end{bmatrix}.$$

Find orthogonal bases for:

(a) Null(A)

(b) Null(A^T)

(c) Col(A)

(d) Col(A^T)

6. Suppose A is an $n \times n$ real symmetric matrix.

(a) Prove that the eigenvalues of A are real, and the corresponding eigenvectors can be chosen to be real.

(b) Prove that eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

7. Let A be an $n \times n$ real symmetric matrix. Prove that there is an orthonormal basis for $\mathbf{R}^n$ consisting of eigenvectors of A. (You may use the results of the previous exercise.)

8. Let A be a real $m \times n$ matrix. Prove that AA^T and A^TAhave the same nonzero eigenvalues.

9. Let P be a symmetric $n \times n$ matrix satisfying $P^2=P$, and assume that P is neither the zero matrix nor the identity matrix. Let W_1 be the column space of P and W_2 be the null space of P.

(a) Prove that $x \in W_1$ if and only if $Px=x$.

(b) Prove that if λ is an eigenvalue of P, then λ is zero or one.

(c) Prove that $\mathbf{R}^n$ is the direct sum of W_1 and W_2. That is, prove that every $x \in \mathbf{R}^n$ can be written uniquely as $x=y+z$, $y \in W_1$, $z \in W_2$.

(d) Prove that, for each $x \in \mathbf{R}^n$, Px is the vector in W_1 closest to x (in the Euclidean norm).

10. Let A be an $n \times n$ matrix. Prove that eigenvectors corresponding to distinct eigenvalues are linearly independent. That is, prove that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of A, and $x_1, x_2, \ldots, x_k$ are corresponding eigenvectors, then $\{x_1, x_2, \ldots, x_k\}$ is a linearly independent set.

11. Suppose $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$are two different orthonormal bases for a subspace W of

$\mathbf{R}^n$. Suppose further that the scalars a_{ij}, i,j=1,2,3, satisfy

$$v_i = \sum_{j=1}^{3} a_{ji} u_j, \; i = 1,2,3.$$

(a)
: Show that the 3×3 matrix A whose entries are a_{ij}, i,j=1,2,3, is orthogonal.

(b)
: Prove that the matrices $u_1u_1^T+u_2u_2^T+u_3u_3^T$ and $v_1v_1^T+v_2v_2^T+v_3v_3^T$ are equal.

(c)
: Interpret the action of $P=u_1u_1^T+u_2u_2^T+u_3u_3^T$: If $x \in \mathbf{R}^n$, what is the significance of Px?

12.
: Let V be an inner product space, let W be a finite-dimensional subspace of V with basis $\{x_1, x_2, \ldots, x_n\}$, and let v be any vector in V.

  (a)
  : Prove that there is a unique vector $w \in W$ closest to v (the best approximation to v from W).

    ``Closest'' is defined in terms of the norm induced by the inner product.

  (b)
  : Derive the normal equations for computing the best approximation w to v from W.

13.
: Give an example to show that Gaussian elimination without partial pivoting can be unstable in finite precision arithmetic. Show that the use of partial pivoting eliminates the instability in your example. (Hint: The matrix need not be large--a 2×2 matrix will do!)

14.
: Suppose A is a nonsingular $n \times n$ matrix and $x, y, b, c \in \mathbf{R}^n$ satisfy

$$\begin{aligned} Ax &= b, \\ A(x+y) &= b+c. \end{aligned}$$

Give a bound on $\|y\|/\|x\|$ in terms of $\|c\|/\|b\|$.