Real Analysis Qualifying Exam

August 2025

Submit complete solutions for exactly 5 of the following 6 problems.

1. (a) State the definition of "supremum":
   We say that $\sup S = a$ if ...

   (b) Let S and T be nonempty subsets of $\mathbb{R}$, with $S \subset T$.
   Prove that $\sup S \leq \sup T$.
   (Note: one or both of the suprema might be infinite.)

2. (a) Prove that all bounded monotone sequences converge.

   (b) Prove that every sequence has a monotonic subsequence.

   (c) State and prove the Bolzano-Weierstrass theorem.

3. (a) Let f be a real-valued function on (a, b), with $x_0 \in (a, b)$.
   Complete the definition: "$f(x)$ is continuous at x_0 if ..."

   (b) Prove that the function $g(x) = \frac{1}{1-x}$ is continuous on $(0, 1)$.

   (c) Let f be a real-valued function on (a, b).
   Complete the definition: "$f(x)$ is uniformly continuous on (a, b) if ..."

   (d) Prove that the function $g(x) = \frac{1}{1-x}$ is not uniformly continuous on $(0, 1)$.

4. State the Mean Value Theorem. Use it to prove the inequalities

$$\frac{h}{1+(x+h)^2} < \arctan(x+h) - \arctan x < \frac{h}{1+x^2}$$

   for all positive x and h.

5. Let $f : (a, b) \to \mathbb{R}$ be a continuous function, let $c \in (a, b)$, and define $F : (a, b) \to \mathbb{R}$ by

$$F(x) = \int_c^x f(t)\, dt.$$

   Prove that F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$.

6. Prove that convergent sequences are Cauchy sequences.