Ordinary Differential Equations
Comprehensive Exam

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Complete any 3 questions from Part I and any 3 questions from Part II of the exam. Start each question on a new page. Label the pages and problems clearly. Indicate clearly which questions you want to have graded. Calculators are NOT ALLOWED.

Part I

1. Use linearization to classify the origin as a sink, source or saddle:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1\\ \dot{x}_2 &= kx_1 - x_2 - x_1x_3, \quad k \neq 1,\\ \dot{x}_3 &= x_1x_2 - x_3.\end{aligned}$$

2. Find the critical points of the system below and use Lyapunov functions to determine the stability of these critical points.

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_1x_2^2\\ \dot{x}_2 &= -2x_1^2x_2 - x_2^3.\end{aligned}$$

   HINT: try $V(x_1, x_2) = x_1^2 + cx_2^2$ for some $c > 0$.

3. Show that the ODE system

$$\begin{aligned}x' &= -y + x(8 - 2x^2 - 2y^2)^2\\ y' &= x + y(8 - 2x^2 - 2y^2)^2\end{aligned}$$

   has a limit cycle. Discuss the stability of the limit cycle. Sketch the phase portrait for this system.

4. State and prove the classical (Picard iterate based) existence and uniqueness theorem for Initial Value Problems. Make sure you clearly define all terms and hypotheses.

Part II

1. The Bogacki-Shampine (BS) scheme is given by the Butcher table

$$\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
3/4 & 0 & 3/4 & & \\
1 & 2/9 & 1/3 & 4/9 & \\
\hline
 & 2/9 & 1/3 & 4/9 & 0 \\
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}$$

   (a) How many stages does this scheme have?

   (b) Calculate the order r of the scheme given by the coefficients immediately below the horizontal line. The row two below the horizontal line is a scheme of order $r-1$.

   (c) Write pseudo-code for one BS step. Explain inputs/outputs and parameters.

2. Consider the following method to solve $y'(t) = f(t, y(t))$:

$$y_{n+1} = y_n + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf(t_{n+1}, y_n + hf(t_n, y_n)).$$

   (i) Find the expression for the stability function

$$R(z) = \left|\frac{y_{n+1}}{y_n}\right|$$

   for this method. Plotting the stability region is not required.

   (ii) Is this method A-stable? Justify your answer.

   (iii) What is the order of accuracy of this method?

3. For the solver

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(y_{n+2})$$

   for the IVP $y' = f(y)$ with $y(0) = 0$.

   - Compute the order.
   - Explain how to compute the stability region.
   - Explain why you would need to do something to start your code.
   - Explain why it is difficult to change step size in a multi-step scheme like this.

4. Black box ODE solvers test if the IVP $y' = f(y)$ with $f : \mathbb{R}^m \to \mathbb{R}^m$ $y(0) = y_0$ is stiff at $t = 0$.

   - Explain what makes a system of ODEs $y' = f(y)$ stiff?
   - Explain how to tell if the linear ODEs $y' = Ay$ defined by $A \in \mathbb{R}^{n\times n}$ is stiff?
   - Discuss a heuristic to detect stiffness from eigenvalues of the derivative $Df(y_0) \in \mathbb{R}^{m\times m}$.