Number Theory Comprehensive Examination
Fall 2025

Complete six problems from the list below. Justify all answers and show all work. If you start on more than six problems, clearly indicate which problems you wish to be graded by thoroughly striking through the other problems.

1. Fix $n > 1$ and let ζ_n be a primitive n-th root of unity. Show that

$$\sum_{d:gcd(d,n)=1} \zeta_n^d = 0.$$

2. Let integers a and b have the properties $a, b > 1$, $a \nmid b$, $b \nmid a$. Show that $c := \frac{\mathrm{lcm}(a,b)}{\gcd(a,b)}$ is an integer with the property that $c \nmid a$, $c \nmid b$.

3. Fix an integer a. Show that the only solution to the system of congruences which holds that $x \equiv a \pmod{p}$ for *all* primes p is $x = a$.

4. Suppose one knows that $n = pq$ is a product of two distinct primes p and q. Show that, if one knows $\phi(n)$, then one can easily calculate p and q as the roots of $x^2 - (n - \phi(n) - 1)x + n$.

5. State the law of quadratic reciprocity. Determine whether 105 is a quadratic residue modulo the prime 107.

6. Let $n \equiv 3 \pmod{4}$. Show that the product of all quadratic nonresidues modulo n is a quadratic nonresidue.

7. State the pentagonal number theorem for partitions. Taking as accepted that the partition function $p(n)$ is a nondecreasing function on $\mathbb{Z}$, show that $p(n) - p(n-1) - p(n-2) < 0$ for all $n \geq 5$.

8. Show the q-series identity

$$\prod_{n=1}^{\infty}(1 + q^n) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n}.$$

9. Show that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ has a singularity at $s = 1$, or, in more classical language, that the harmonic series diverges.