

Mathematical Statistics Qualifying Exam, Fall 2025

Department of Mathematical Sciences

Michigan Technological University

Direction:

- Partial credit will be awarded as long as you show all your work and correctly solve parts of the problem. Points will be deducted if you fail to show your work or if some parts are incorrect, even if the final answer is correct.
- You may use one part of the problem to solve another, even if you were unable to solve the previous part (you may assume any necessary values).
- Please write your answers on one side of the page only and start the solution to each problem on a new page.
- A calculator may be used.

1. Check to see whether the continuous random variables X and Y are independent for each of the following joint probability density functions. Justify your answer—a simple “Yes” or “No” answer is not sufficient.

$$\begin{aligned}
 \text{(a)} \quad f(x, y) &= \begin{cases} 2x^{-3}e^{1-y}, & 1 < x, y < \infty, \\ 0, & \text{otherwise,} \end{cases} \\
 \text{(b)} \quad f(x, y) &= \begin{cases} \frac{4}{5}(1 + xy), & 0 < x, y < 1, \\ 0, & \text{otherwise,} \end{cases} \\
 \text{(c)} \quad f(x, y) &= \begin{cases} 3x^2y^{-3}, & 0 < x < y < 1, \\ 0, & \text{otherwise,} \end{cases} \\
 \text{(d)} \quad f(x, y) &= \frac{1}{\sqrt{3}\pi} \exp \left\{ -\frac{2}{3}(x^2 - xy + y^2) \right\} \text{ for all } x, y \in \mathbb{R}.
 \end{aligned}$$

2. A customer service center receives calls continuously throughout the day. The time between consecutive calls follows an exponential distribution with an average of 20 minutes between calls.

- (a) Let Y be the waiting time (in minutes) until the next call arrives. Write down the probability density function (pdf) of Y .

Hint: Recall that the pdf of an exponential distribution is $f(y) = \lambda e^{-\lambda y}$ if $y \geq 0$, and 0 otherwise, for some $\lambda > 0$.

- (b) Compute the mean and variance of Y .
- (c) The manager decides to check the call log every minute to see if a call has arrived during that interval. Let X be the number of one-minute intervals the manager observes **until the first call is detected** during an interval. That is, if the first call arrives during the first minute of observation, then $X = 1$. Assume the number of calls in non-overlapping intervals is independent. Find the probability mass function (pmf) of X .
- (d) Find the mean and variance of X .

Hint: For any $|r| < 1$, $\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2}$ and $\sum_{k=1}^{\infty} k^2 r^{k-1} = \frac{1+r}{(1-r)^3}$.

- (e) Compare the means and variances of X and Y . Are they similar or noticeably different? Explain your intuition for why this is the case.

3. The lifetime (in hours) of Y of an electronic component is a random variable with the probability density function

$$f(y) = \begin{cases} \frac{1}{50}e^{-y/50}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the probability that the component will operate for 100 hours without failure.
- (b) Given that the component has already operated for 100 hours, find the probability that it will operate for another 100 hours without failure.

- (c) Suppose three such components operate independently in a piece of equipment. The equipment fails if at least two components fail. Find the probability that the equipment operates for at least 100 hours without failure.
4. A box contains 3 black balls, 2 white balls, and 2 red balls. You randomly select 4 balls without replacement. Define:
- X = number of black balls in the selection,
 - Y = number of red balls in the selection.
- (a) How many different groups of 4 balls can be selected from the box?
- (b) How many ways are there to select exactly 2 black balls and 1 red ball among the 4 balls?
- (c) Find the joint probability mass function of X and Y . Present your results in a table.
- (d) Given that exactly one red ball is selected, find the conditional probability mass function of the number of black balls in the selected sample of 4 balls.
5. Let $X_1, X_2, \dots, X_{n_1}$ be a random sample from a normal distribution with mean μ and variance σ^2 , and let $Y_1, Y_2, \dots, Y_{n_2}$ be an independent random sample from a normal distribution with mean 6μ and variance $4\sigma^2$. Further, let

$$\bar{X} = \frac{1}{n_1} \sum_{j=1}^{n_1} X_j, \quad S_X^2 = \frac{1}{n_1 - 1} \sum_{j=1}^{n_1} (X_j - \bar{X})^2, \quad \bar{Y} = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j, \quad S_Y^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} (Y_j - \bar{Y})^2$$

denote the sample means and sample variances of the two random samples. Assume that μ and $\sigma > 0$ are unknown parameters. There may be several correct answers to the following questions; please present only one answer for each.

- (a) Find a transformation of $\bar{X}$ and $\bar{Y}$ that has a standard normal distribution. Your answer should not involve μ , but may involve σ .
- (b) Find a transformation of S_X^2 and S_Y^2 that has a χ^2 distribution. Your answer should not involve μ , but may involve σ . Make sure that both S_X^2 and S_Y^2 are included in your transformation. Also, specify the degrees of freedom associated with the χ^2 distribution.
- (c) Find a transformation of $\bar{X}$, $\bar{Y}$, S_X^2 , and S_Y^2 that has a Student's t distribution. Your answer should not involve any unknown parameters. Justify your answer by stating the definition of the t distribution and arguing that your transformation meets the requirements.

6. Let $X_1, X_2, \dots, X_n$ be a random sample from a Poisson distribution with mean λ , and let $Y_1, \dots, Y_n$ be an independent random sample from a Poisson distribution with mean 2λ . Recall that this implies

$$\mathbb{E}[X_j] = \text{Var}[X_j] = \lambda \quad \text{and} \quad \mathbb{E}[Y_j] = \text{Var}[Y_j] = 2\lambda$$

Further, let

$$\hat{\lambda}_1 = \frac{1}{n+1} \sum_{j=1}^n X_j, \quad \hat{\lambda}_2 = \bar{X}, \quad \hat{\lambda}_3 = \frac{1}{2}\bar{Y}, \quad \hat{\lambda}_4 = \frac{1}{3}(\bar{X} + \bar{Y})$$

be different choices of point estimators of λ , where $\bar{X} = (X_1 + \dots + X_n)/n$ and $\bar{Y} = (Y_1 + \dots + Y_n)/n$ are the sample means of the two respective samples.

- Determine the bias and mean square error (MSE) of $\hat{\lambda}_1$. You do not need to simplify either expression.
 - Determine the bias and MSE of $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\lambda}_4$. Which of these estimators would you recommend as being the “best” at estimating the parameter λ ?
 - Derive a 95% large-sample confidence interval for λ constructed around $\hat{\lambda}_4$. Recall that the 0.975-quantile from the standard normal distribution is approximately 1.96.
7. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x) = \frac{2\sqrt{\theta}}{\sqrt{\pi}} e^{-\theta x^2}, \quad x > 0,$$

where $\theta > 0$ is an unknown parameter. You may assume that

$$\mathbb{E}[X_j] = \frac{1}{\sqrt{\pi\theta}}, \quad \mathbb{E}[X_j^2] = \frac{1}{2\theta}.$$

- Determine a simplified expression for the likelihood function without using product notation (such as $\prod_{i=1}^n$).
- Let $\hat{\theta}_1$ be the method-of-moments estimator of θ based on the first moment. Find an algebraic expression for $\hat{\theta}_1$.
- Prove that $\hat{\theta}_1$ is a consistent estimator of θ , citing any well-known results or named theorems you use.
- Let $\hat{\theta}_2$ be the maximum likelihood estimator of θ . Determine an algebraic expression for $\hat{\theta}_2$.
- Which of $\hat{\theta}_1$ and $\hat{\theta}_2$ are sufficient for θ ? (One, both, or neither?) Justify your answer.
- Determine the Fisher information of θ associated with this sample. Using the known asymptotic normality of the MLE, find a transformation of $\hat{\theta}_2$ that converges in distribution to a standard normal distribution. Note that your transformation may involve θ .

8. Let Y be a single random variable with probability density function

$$f_Y(y|\theta) = \begin{cases} \theta y^{\theta-1}, & 0 \leq y \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Note that the cumulative distribution function and quantile function of Y are

$$F_Y(y) = \begin{cases} 0, & y \leq 0, \\ y^\theta, & 0 < y \leq 1, \\ 1, & y > 1, \end{cases} \quad \text{and} \quad F_Y^{-1}(p) = p^{1/\theta},$$

respectively.

- (a) Using the Neyman-Pearson lemma, determine the rejection region of the most powerful level 0.10 test for the simple hypotheses

$$H_0: \theta = 1 \quad \text{against} \quad H_1: \theta = 2.$$

Further, determine the power of this hypothesis test.

- (b) Find the rejection region of the uniformly most powerful level 0.10 test for

$$H_0: \theta = 1 \quad \text{against} \quad H_1: \theta > 1.$$

- (c) Prove that $U = -\theta \log(X)$ is a pivotal quantity for θ by demonstrating that it follows an exponential distribution with mean 1. Using this pivotal quantity, derive a 90% confidence interval for θ .

Hint: Note that the quantile function of $U \sim \text{Exp}(1)$ is

$$F_U^{-1}(p) = -\log(1 - p), \quad 0 < p < 1.$$