Qualifying Examination
Linear Algebra

Fall 2025

Instructions:

- There are two parts in the exam. Solve only three out of the four problems in each part. Indicate which three problems you have chosen to solve. Only first three solved problems in each part will be graded.
- Start each problem on a new sheet of paper.
- Define all variables that are introduced in solving the problems.
- No calculators or electronic devices are allowed.
- Justify all answers and show all work.
- Each problem carries the same weight.

- Part 1. Solve three out of the four problems.

  1. Let $V = \mathbb{R}^4$, and consider the subspace W spanned by the vectors $\boldsymbol{v}_1 = (1, 2, -1, 0)$, $\boldsymbol{v}_2 = (0, 1, 2, 3)$, and $\boldsymbol{v}_3 = (2, 5, 0, 3)$.
     (a) Determine if the vectors, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$,$\boldsymbol{v}_3$ are linearly independent.
     (b) Find a basis and the dimension of W.
     (c) Find a vector in $\mathbb{R}^4$ that is orthogonal to W.
  2. Consider the matrix $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$.
     (a) Find eigenvalues and eigenvectors of A.
     (b) Compute A^{50} using the diagonalization of A.
     (c) Is A diagonalizable? Justify your answer.
  3. Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, which is symmetric.

(a) Compute the eigenvalues and the corresponding eigenvectors of A. Show that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

(b) Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

4. Find the least-squares solution of $A\boldsymbol{x} = b$ where:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}.$$

(a) Set up and solve the normal equations $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$.

(b) Verify that the solution minimizes the error $||A\boldsymbol{x} - \boldsymbol{b}||$.

- Part 2. Solve three out of the four problems.

1. Let A be a positive definite real matrix. Show that

(a) There exists an invertible real matrix B such that $A = B^TB$.

(b) There exists a positive definite real matrix B such that $A = B^2$.

2. Let V be an inner product space over a field $\mathcal{F}$.

(a) Let $\{u_1, \cdots, u_m\}$ be a basis of V, Show that the Gram matrix $G = (\langle u_i, u_j\rangle)_{m\times m}$ is invertible.

(b) Let U be another inner product space over the field $\mathcal{F}$, and $T : U \to V$ be a linear operator. Show that kernel$(T) \cap$ range$(T^\star) = \{\vec{0}\}$, where $T^\star$ is the adjoint operator of T, i.e. $\langle T(u), v\rangle = \langle u, T^\star(v)\rangle$.

3. Let $\mathcal{F}$ be a number field, $A \in \mathcal{F}^{m\times n}$, and $B \in \mathcal{F}^{n\times k}$. Show that $\text{rank}(A) + \text{rank}(B) \leq n + \text{rank}(AB)$.

4. Given finite dimensional linear spaces over a number field $\mathcal{F}$: $V_0 = \{0\}, V_1, \cdots, V_n, V_{n+1} = \{0\}$. Let $\phi_i : V_i \to V_{i+1}$ be linear transforms satisfying kernal(ϕ_{i+1}) = range(ϕ_i), $i = 0, \cdots, n-1$. Show that $\displaystyle\sum_{i=1}^{n}(-1)^i\text{dim}(V_i) = 0$.