Design Theory Comprehensive Exam
Fall 2025

Instructions: Complete six problems from the list below. Justify all answers and show all work. If you start on more than six problems, clearly indicate which problems you wish to be graded by thoroughly striking through the other problems.

1. Let S be a set of size v and let T be a set of 3-element subsets of S. Furthermore, suppose that

   (a) each pair of distinct elements of S belongs to at least one triple in T, and

   (b) $|T| \leq \frac{v(v-1)}{6}$.

   Show that (S, T) is a Steiner triple system.

2. The complete directed graph of order n, denoted by D_n, is the graph with n vertices in which each pair of distinct vertices are joined by two directed edges (in opposite directions). We will denote the directed edge from a to b by (a, b) and the directed edge from b to a by (b, a). A directed triple is a collection of three directed edges of the form $\{(a, b), (b, c), (c, a)\}$ where a, b, and c are distinct. We will denote this directed triple by any cyclic shift of (a, b, c). A Mendelsohn triple system of order n (MTS(n)) is a pair (S, T), where T is an edge disjoint collection of directed triples which partitions the edge set of D_n with vertex set S.

   (a) Show that a necessary condition of the existence of an MTS(n) is $n \equiv 0$ or $1 \pmod 3$.

   (b) Find an MTS(4). (A picture of D_4 and the triples in the MTS(4) is fine.)

3. Use the following Skolem sequence of order 4 to find a 1-factorization of K_{10}: $(1, 1, 3, 4, 2, 3, 2, 4)$.

4. A Latin square is idempotent if, for every element "i", the entry in the cell (i, i) is also "i". This means the main diagonal of the square consists of the numbers $1, 2, \ldots, n$ in that order. Show that if $n \geq 3$ and $\{L_1, \ldots L_t\}$ is a set of MOLS(n), all of which are idempotent, then $t \leq n - 2$.

5. Construct a regular pairwise balanced design on six points that contains exactly four blocks of size three. (Note: The design may consist of more than four blocks.)

6. Find a $(15, 7, 3)$-difference set in $(\mathbb{Z}_{15}, +)$.

7. (a) Construct a resolvable $(6, 2, 1)$-BIBD on the points $\{\infty, 0, 1, 2, 3, 4\}$.

   (b) Using the blocks of the design from part (a), construct a 3-$(6, 4, 3)$ design.

8. Show that in any resolvable Steiner triple system (a.k.a. Kirkman triple system)

   (a) the number of triples in each parallel class is $\frac{v}{3}$, and

   (b) the number of parallel classes is $\frac{v-1}{2}$.