Partial Differential Equation Comprehensive Exam

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PART I - Analytic Section. Complete ANY 3 out of 4 questions. Start each question on a new page. Label the pages and problems clearly.

1. Let $\Omega = (0, \pi) \times (0, \pi)$. Use separation of variables to solve the boundary value problem

$$\begin{aligned} \Delta u = 0 \qquad & \text{for } (x,y) \in \Omega, \\ u_x(0,y) = 0 = u_x(\pi, y), \ & \text{for } 0 < y < \pi, \\ u(x,0) = 0, u(x,\pi) = g(x), \ & \text{for } 0 < x < \pi. \end{aligned}$$

2. State and prove the weak and strong maximum principle of the Lapace equation on an open, bounded and connected doamin in $\mathcal{R}^2$.

3. Let Ω be a smooth, bounded domain in R^3.

   3a) For a C^2 solution $u(x,t)$ of the wave equation $u_{tt} = c^2 \Delta u$ for $x \in \Omega, t > 0$, define the energy to be

$$E_\Omega(t) = \frac{1}{2} \int_\Omega (u_t^2 + c^2 |\nabla u|^2) dx.$$

   If u satisfies the boundary condition $u(x,t) = 0$ for $x \in \partial\Omega$ and the initial conditions $u(x,0) = 0 = u_t(x,0), x \in \Omega$, then show that $E_\Omega(t)$ is constant.

3b) Use the result of **(3a)** to show uniqueness of the solution for the nonhomogeneous wave equation $u_{tt} = c^2\Delta u + f(x,t)$ in a smooth, bounded domain $\Omega \subset R^3$ with the Dirichlet boundary condition $u(x,t) = g(x,t)$ on $\partial\Omega$ and the initial conditions $u(x,0) = h(x), u_t(x,0) = m(x), x \in \Omega$.

4. A square two-dimensional plate of side length a is heated to a uniform temperature $U_0 > 0$. Then at time $t = 0$ all four sides are reduced to zero temperature. Assume that the top and bottom of the plate are insulated so that the heat flow can be restricted to two dimensions.

   4a) Introduce the problem (the PDE and all the accompanying conditions) that the temperature $u(x,y,t)$ satisfies.

   4b) Solve the problem from **(4a)** using the Eigenfunction expansion.

PART II - Numeric Section. Complete ANY 3 out of 4 questions. Start each question on a new page. Label the pages and problems clearly.

1. Consider the IVP

$$u_t + u_x = \epsilon u_{xx}, \quad u(x,0) = f(x) \tag{0.1}$$

   where ϵ is a positive constant.

   (a) Design an explicit finite difference scheme to solve the above problem with local truncation error analysis.

   (b) Design an implicit finite difference scheme with detailed local truncation error analysis.

   (c) Use Von Neumann analysis to derive the proper time step constraint needed to ensure the stability of the two numerical schemes designed previously.

2. (a) Find the element stiffness matrix for the problem

$$-u'' = f \quad 0 < x < 1, \quad u(0) = u(1) = 0, \tag{0.2}$$

if piecewise quadratic functions are used for the finite element Galerkin method.

(b) Determine the corresponding global stiffness matrix in the case of a uniform subdivision. Compare the resulting equation with that obtained by central finite difference approximation of the equation (0.2).

(c) Propose a method to solve the linear system you obtained and justify your approach.

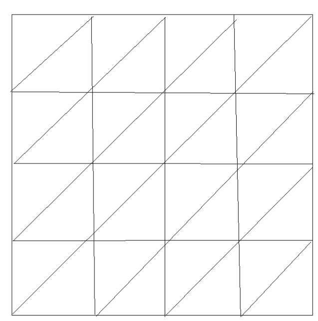
3. Consider solving Poisson's equation on $\Omega = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$:

$$-\Delta u = f(x, y) \tag{0.3}$$

$$u = 0, \text{ for } (x, y) \in \partial\Omega \tag{0.4}$$

(a) Set up a partition of the domain and write down the central finite difference scheme for solving the Laplacian problem.

(b) Set up a finite element method with piecewise linear bases to solve the probelm. Write out the final linear system for solving based on a mesh of this arrangement.



Use the number and shape of elements as indicated in this graph.

(c) What are the similarities and differences between the two schemes? Discuss the accuracy of each method.

4. Let Ω be a square with side 1, Γ_1 is the top and left boundary, Γ_2 is the right and bottom boundary.

(a) Give a variational formulation of the following problem

$$\Delta u = f \quad \mathbf{x} \in \Omega, u = u_0 \quad \mathbf{x} \in \Gamma_1, \quad \frac{\partial u}{\partial n} = g, \quad \mathbf{x} \in \Gamma_2. \tag{0.5}$$

(b) Formulate a finite element method for this problem, specifying the solution space.

(c) When $u_0 \equiv 0$, determine the stiffness matrix if rectangular mesh and bilinear elements are used for uniform subdivision $h = 1/4$.