ALGEBRA QUALIFYING EXAM

Fall 2025

Instructions. Solve 2 of the 3 problems from each of the four sections. Clearly indicate which problems you wish to have graded. For full credit you must provide justification of all your answers.

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Group Theory

1. Fix a group G, and define the map $\varphi : G \longrightarrow \mathsf{Aut}(G)$ by $\varphi(g) = \gamma_g$ (conjugation by the element g). Prove that:

a) φ is a group homomorphism.

b) The kernel of φ is the center of G.

c) The image of φ is a normal subgroup of $\mathsf{Aut}(G)$.

2. Let G be a group of order pq where p and q are distinct primes with $p < q$ and p does not divide $q - 1$. Use Cauchy's Theorem and Lagrange's Theorem to show that G is cyclic.

3. Describe the action of the symmetric group S_4 on the set of subsets of size 2 from $\{1, 2, 3, 4\}$, and compute the number of orbits using Burnside's Lemma. (Be sure to use Burnside's Lemma; do not simply list the orbits).

Linear Algebra

4. Let k be a field, and V and W two vector spaces of dimension 2025 over k. Show that V and W are isomorphic.

5. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, compute its characteristic polynomial and trace, and determine if it is diagonalizable.

6. Let $V = \mathbb{R}^4$ be a vector space over $\mathbb{R}$, and consider the set $S = \{(1,0,1,0),(0,1,1,1)\} \subset V$.

a) Prove that S is linearly independent.

b) Extend S to a basis for $\mathbb{R}^4$.

c) Define a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$T(x_1,x_2,x_3,x_4) = (x_1+x_2, x_2-x_3, x_3+x_4, x_4-x_1).$$

Find the matrix representation of T with respect to the basis you constructed in part (b).

Ring Theory

For the next problems assume all rings are commutative with identity.

7. Let A be a *Boolean* ring (i.e., all elements $a \in A$ satisfy $a = a^2$).

a) Determine all units and all zero-divisors of A.

b) Show that every finitely generated ideal of A is principal. (*HINT*: Prove that the ideals (r, s) and $(r + s - rs)$ coincide, for any $r, s \in A$.)

8. Let S be a ring whose proper ideals are all prime. Show that S is a field.

9. State and prove the First Isomorphism Theorem for rings, and apply it to show that

$$\mathbb{Z}[x]/(x^2 + 1) \cong \mathbb{Z}[i]$$

Fields

10. Let D be a domain that contains a field k. Prove that:

a) D is a k-vector space.

b) If D is finite dimensional over k, then D is a field.

11. Let $\alpha = \sqrt{2} + \sqrt{3} \in \mathbb{R}$.

a) Prove that α is algebraic over $\mathbb{Q}$ and find its minimal polynomial.

b) Determine the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$.

c) Determine a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, and express α^3 and α^{-1} as linear combinations of those basis elements.

12. Determine all possible field homomorphisms from K to F, where:

a) $K = \mathbb{C}$ and $F = \mathbb{R}$.

b) $K = \mathbb{R}$ and $F = \mathbb{Q}$.

c) $K = F = \mathbb{Q}$.