PhD Comprehensive Examination
Partial Differential Equations

August 2026

Instructions

- Please solve three of the four problems in Section I (Theory).
- Please solve three of the four problems in Section II (Numerical Approximations).
- Start each problem on a new sheet of paper; label the problem number clearly.
- Define all variables that are introduced in solving the problems.
- No calculators or electronic devices are allowed.
- Justify all answers and show all work.
- Each problem carries the same weight; individual subproblems may be weighted differently.

I: Theory

1. (a) Define weak solutions of Burger's equation

$$u_t + uu_x = 0. \tag{1}$$

   (b) Explain how the method of characteristics gives formulas for smooth solutions. Explain when and how this method fails.

   (c) Find a weak solution to eq. (1) with the initial condition

$$u(0,x) = \begin{cases} 1 & x \leq 0 \\ 2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \tag{2}$$

   (d) Explain how eq. (1) is connected with the conservation law

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0. \tag{3}$$

   (e) Find a weak solution to eq. (3) for small time > 0, with the initial conditions given in eq. (2).

2. Write down the heat equation and then state and prove a maximum principle for your PDE. Make sure you define all the terms and notation you use. Ensure your assumptions and definitions are complete.

3. Consider the PDE:

$$\begin{aligned} \Delta u(\mathbf{x}) &= f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega \end{aligned}$$

   where $\Omega = \{\mathbf{x} = (x,y,z) \subseteq \mathbb{R}^3 : z > 0, \|\mathbf{x}\|_2 < a\}$. Construct an appropriate Green's function for this problem.

4. Let $f \in C^1(\mathbb{R})$, let f and f' be bounded functions, and consider the Cauchy problem,

$$\begin{aligned} & u_t + (u + u^2)u_x = 0, \qquad x \in \mathbb{R}, \qquad t > 0 \\ & u(0,x) = f(x), \qquad x \in \mathbb{R}. \end{aligned}$$

   Prove that a continuously differentiable solution, $u(t,x)$, exists and is unique for $x \in \mathbb{R}$ and $t \in [0, t_\star)$ for some $t_\star > 0$.

II: Numerical Approximations

5. Consider the heat equation

$$u_t = c\, u_{xx}$$

with homogeneous Dirichlet boundary conditions.

   (a) Show that the following finite difference scheme,

   $$\frac{U_j^{n+1} - U_j^n}{\Delta t} = c\, \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}$$

   is first-order accurate in time and second-order accurate in space.

   (b) Use von Neumann analysis to show that this finite difference method is unconditionally stable.

   (c) State the Lax Equivalence Theorem and use it to conclude that the finite difference method converges.

6. Denote T as the standard reference tetraheron satisfying

$$T = \{(x, y, z) \geq 0 \mid x + y + z \leq 1\}.$$

   (a) Discuss the affine transformations that map the reference tetrahedron to a general tetrahedron, $\hat{T}$, with vertices $\{(x_i, y_i, z_i)\}$, $i = 1, 2, 3, 4$.

   (b) Use your transformations to find an expression for $|\hat{T}|$, the volume of the $\hat{T}$. Justify your answer.

7. Carefully describe a (simple) numerical approximation scheme for Burger's equation

$$u_t + u\, u_x = 0 \quad u(x, 0) = u_0(x) \geq 0.$$

Explain the role of any parameters and/or evaluation choices you make.

8. (a) Let u denote the temperature. Write down the boundary value problem for equilibrium heat conduction on a quarter circle, $x^2 + y^2 < 1$ with $x > 0$ and $y > 0$. Assume:

   - the presence of a heat source f;
   - insulation on the curved boundary;
   - the temperature is held fixed on the straight portions of the boundary.

   (b) Describe a triangular mesh for this domain. Quantify any approximations in this meshing in terms of h, the maximum side length.

   (c) Explain how to construct standard piecewise linear elements on this mesh. Explain how accurately elements of the resulting function space approximate smooth functions.

   (d) Derive a weak form for your BVP explain how boundary conditions are incorporated.

   (e) Describe any issues at the corners.

   (f) Describe how you would go about implementing your scheme.