PhD Comprehensive Examination
Ordinary Differential Equations

August 2026

Instructions

- Please solve three of the four problems in Section I (Theory).
- Please solve three of the four problems in Section II (Numerical Approximations).
- Start each problem on a new sheet of paper; label the problem number clearly.
- Define all variables that are introduced in solving the problems.
- No calculators or electronic devices are allowed.
- Justify all answers and show all work.
- Each problem carries the same weight; individual subproblems may be weighted differently.

I: Theory

1. State and prove the classical Cauchy existence and uniqueness theorem for

$$y' = f(t, y), \quad y(t_0) = y_0.$$

   Demonstrate non-uniqueness in a simple ODE which does not satisfy the hypothesis.

2. Let f and $\frac{\partial f}{\partial y}$ be continuous and bounded, and consider the ODE

$$y' = f(t, y). \tag{1}$$

   Suppose that $u_0(t)$ is the solution of eq. (1) that passes through the point (t_0, y_0), and $u_1(t)$ is the solution of eq. (1) that passes through the point (t_1, y_1). Assume that both $u_0(t)$ and $u_1(t)$ exist on some interval $\mathcal{I} = (\tau_1, \tau_2)$.

   Show that for each $\epsilon > 0$, there exist $\delta > 0$ such that if $|t - \hat{t}| < \delta$ and $\|y_0 - y_1\| < \delta$, then $\|u_0(t) - u_1(\hat{t})\| < \epsilon$ for $t, \hat{t} \in \mathcal{I}$. You may invoke Gronwall's inequality without proof.

3. Give the general solution of $y' = A\,y$ where

$$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

   Describe and sketch the behavior of solutions near the equilibrium point. Show how to derive the variation of parameters solution of

$$y' = A\,y + b(t), \quad y(t_0) = y_0.$$

4. (a) State the Poincaré–Bendixson theorem.

   (b) The system of ODEs

$$\begin{aligned} x' &= x(1 - x^2 - y^2) - y \\ y' &= y(1 - x^2 - y^2) + x, \end{aligned}$$

   can be transformed using polar coordinates:

$$\begin{aligned} r' &= r(1 - r^2) \\ \theta' &= 1. \end{aligned}$$

   Show that $\frac{1}{2} \leq r \leq 2$ is a trapping annulus, and make appropriate conclusions using the Poincaré–Bendixson theorem.

II: Numerical Approximations

5. (a) What does it mean for a numerical integrator to be A-stable?
   (b) Show that the implicit Euler method is A-stable.
   (c) Show that the trapezoidal method is A-stable.

6. Pretend that you had to give a brief guest lecture in a numerical ODE class. Explain stability regions and include the stability region of an integrator of your choice.

7. Consider a linear multi-step scheme of the form

$$y_{n+1} = a_1\, y_n + a_2\, y_{n-1} + h\,(b_0\, f(t_{n+1}, y_{n+1}) + b_1\, f(t_n, y_n))$$

   (a) Explain the highest order of accuracy this scheme can attain. (hint: justify your answer using some general "rule of thumb").
   (b) Derive the coefficients a_1, a_2, b_0, b_1 so that the constructed scheme reaches this order of accuracy.
   (c) Determine whether or not the obtained scheme satisfies the root condition.

8. For the IVP $y' = f(y)$, with $y(0) = y_0 \in \mathbb{R}^n$.

   (a) Write a fixed step-size explicit Euler scheme for the IVP.
   (b) Write a fixed step-size implicit Euler scheme for the IVP.
   (c) Compare and contrast the accuracy, stability, and implementation of the explicit and implicit schemes.
   (d) When is the additional complexity of implicit Euler appropriate?
   (e) What happens if you try to run a variable step size explicit Euler code on an inappropriate problem?

   Make sure you define all terms you use and explain how you would solve any needed equations.