Comprehensive Exam Fall 2026

Mathematical Statistics

Instructions

- There are two parts in the exam. Solve only three out of the four problems in each part. Indicate which three problems you have chosen to solve. Only first three solved problems in each part will be graded.
- Start each problem on a new sheet of paper; label the problem number clearly.
- Define all variables that are introduced in solving the problems.
- No calculators or electronic devices are allowed.
- Justify all answers and show all work.

Part 1.

1. Suppose that the joint pdf of (X_1, X_2, X_3) is given by

$$f(x_1, x_2, x_3) = (2\pi)^{-3/2}\exp\left\{-\frac{1}{2}[x_1^2 + x_2^2 + x_3^2]\right\}\left(1 + x_1x_2x_3\exp\left\{-\frac{1}{2}[x_1^2 + x_2^2 + x_3^2 - 3]\right\}\right)$$

for $-\infty < x_1, x_2, x_3 < \infty$.

a) Show that X_1, X_2, X_3 are dependent random variables.

b) Show that each pair (X_1, X_2), (X_2, X_3) and (X_1, X_3) is distributed as a bivariate normal random vector.

2. Suppose that $X_1, \cdots, X_n$ are iid $N_2(\mu, \Sigma)$ where $\mu \in \mathcal{R}^2$ and Σ is a 2×2 positive definite matrix, $n \geq 2$. Let us denote $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $W = \sum_{i=1}^{n}(X_i - \bar{X})\,(X_i - \bar{X})'$. Prove the following sampling distributions:

a) $\bar{X}$ is distributed as $N_2(\mu, n^{-1}\Sigma)$.

b) $n(\bar{X} - \mu)'\Sigma^{-1}(\bar{X} - \mu)$ is distributed as χ_2^2.

3. Suppose that $Y_1, \cdots, Y_n$ are independent random variables where Y_i is distributed as $N(X_i\beta, 1)$ with $(X_i)_{1\times p}$ being the fixed vector, $\beta_{p\times 1}$ being the unknown parameter vector, for $i = 1, \cdots, n$. Assume the prior of $\beta \sim N_p(b,\ C)$, with b being the known mean vector and C being the known covariance matrix. Derive the posterior joint density of β given $Y_1, \cdots, Y_n$.

4. Suppose that $Y_1, \cdots, Y_n$ are independent random variables where Y_i is distributed as $N(\beta_0 + \beta_1 x_i, \sigma^2)$ with unknown parameters β_0, β_1. Here, $\sigma\ (> 0)$ is assumed known and x_i's are fixed real numbers with $\theta = (\beta_0, \beta_1) \in \mathcal{R}^2$, $i = 1, \cdots, n$, $n \geq 2$. Denote $a = \sum_{i=1}^{n}(x_i - \bar{x})^2$ with $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and assume $a > 0$. Suppose that the MLE's for β_0 and β_1 are respectively denoted by $\widehat{\beta_0}, \widehat{\beta_1}$. Now, consider the following linear regression problems.

a) Write down the likelihood function of $Y_1, \cdots, Y_n$.

b) Show that $\widehat{\beta_1} = \frac{1}{a}\sum_{i=1}^{n}(x_i - \bar{x})\,Y_i$ and $\widehat{\beta_1}$ is normally distributed with mean β_1 and variance σ^2/a.

c) Show that $\widehat{\beta_0} = \bar{Y} - \widehat{\beta_1}\bar{x}$ and $\widehat{\beta_0}$ is normally distributed with mean β_0 and variance $\sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{a}\right]$.

Part 2.

1. Let $Y_1, Y_2, ..., Y_n$ be a random sample from a population with density function

$$f(y \mid \theta) = \begin{cases} \dfrac{3y^2}{\theta^3}, & 0 \le y \le \theta \\ 0, & \text{elsewhere.} \end{cases}$$

a) Show that $Y_{(n)} = \max(Y_1, Y_2, ..., Y_n)$ is sufficient for θ.

b) Show that $Y_{(n)}$ has probability density function

$$f_{(n)}(y \mid \theta) = \begin{cases} \dfrac{3ny^{3n-1}}{\theta^{3n}}, & 0 \le y \le \theta \\ 0, & \text{elsewhere.} \end{cases}$$

c) Estimate θ by the method of moments.

2. Let $X_i, i = 1, 2, ..., n,$ be independent Bernoulli p random variables and let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

a) Show that $\sqrt{n}(Y_n - p) \to N(0, p(1-p))$ in distribution.

b) Show that for $p \ne 1/2, \sqrt{n}(Y_n(1-Y_n) - p(1-p)) \to N(0, p(1-p)(1-2p)^2)$ in distribution.

3. $X_1, ..., X_n$ be a random sample from a $N(0, \sigma_X^2)$, and let $Y_1, ..., Y_n$ be a random sample from a $N(0, \sigma_Y^2)$, independent of the Xs. Define $\lambda = \dfrac{\sigma_Y^2}{\sigma_X^2}$.

a) Find the level α LRT of $H_0 : \lambda = \lambda_0$.

b) Express the rejection region of the LRT in (a) in term of an F random variable.

c) Find a $1-\alpha$ confidence interval for λ.

4. Let $X_1, ..., X_n$ be a random sample from exponential (θ) with the density function of

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \text{ x} \ge 0, \ \theta > 0.$$

a) Find a UMP size α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

b) Find a UMA $1-\alpha$ confidence interval based on inverting the test in part (a). Show that the interval can be expressed as

$$C^*(x_1,\dots,x_n)=\left\{\theta:0\le\theta\le\frac{2\sum x_i}{\chi^2_{2n,\alpha}}\right\}.$$