Recursive Correction - a Novel Technique for Decoupling and Linearization of Navier-Stokes Systems

Alexander E. Labovsky[1]

Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA.

Abstract

We present Recursive Correction - a novel approach to linearization and decoupling of stiff problems; it is based on using defect correction to restore the structure of the differential operator. We use numerical testing to demonstrate that this new method can converge quadratically when linearizing the Navier-Stokes equations. We also test the method on MagnetoHydroDynamic flows: Recursive Correction is shown to improve the quality of the employed decoupling method. Finally, we combine Recursive Correction with the Refactorized Midpoint method for an even more efficient decoupling of MagnetoHydroDynamic flows.

Keywords: recursive correction, defect correction, Refactorized Midpoint, efficient decoupling, NSE, MHD

1. Introduction

In this paper, we propose a new method for linearization of the Navier-Stokes equations and, especially, for an efficient decoupling of the coupled Navier-Stokes systems. The method, which we call Recursive Correction (RC), belongs to the defect correction family. Defect correction methods (DCMs) have been studied in the context of ODEs and PDEs since at least the 1940s, [9]. The interest in the DCMs peaked in 1960s-1970s, with the works of Zadunaisky [10, 11] followed by various groups of researchers, including Frank [12], Frank and Ueberhuber [13], Hemker [14, 15], Böhmer, Hemker and Stetter [16] and others. Since then, many groups have studied defect correction (see, e.g., [26, 27, 25]), including the subfamily of deferred correction methods for improved temporal accuracy, [32, 35, 36].

Defect correction methods have also been used in resolving coupled problems in CFD, including MagnetoHydroDynamics [20, 40, 3] and fluid-fluid interaction [19, 2]. The success of the DCMs in fluid flow modeling has been limited, primarily due to the asymptotic convergence of these methods: it often takes several refinements of the spatial mesh diameter and/or the time step size to observe the claimed convergence rates. Several recent results on the usage of DCMs in CFD, including various applications of defect correction in turbulence modeling, are collected in [1].

Recursive Correction differs from the existing DCMs. While other defect/deferred correction methods use correction to gain more accuracy in approximating the solution of the original continuous problem, the RC is built to restore the discrete (or semi-discrete) differential operator, modified in the defect step for purposes such as linearization or decoupling.

Consider the PDE

$$\frac{\partial}{\partial t}\mathbf{y}(t,\mathbf{x}) = F(t,\mathbf{x},\mathbf{y}(t,\mathbf{x})), \tag{1.1}$$

Email address: `aelabovs@mtu.edu` (Alexander E. Labovsky)

and the Backward Euler method for time discretization (with or without spatial discretization):[1]

$$\frac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = F(t_{n+1}, \mathbf{y}^{n+1}). \tag{1.2}$$

Linearization or decoupling can modify the right-hand side of (1.2) to include the solution at previous time levels. In order to illustrate the RC mechanism, we will write the modified problem as

$$\frac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = \mathcal{F}(t_{n+1}, \mathbf{y}^n, \mathbf{y}^{n+1}). \tag{1.3}$$

The general RC procedure is formalized below.

Algorithm 1.1. *In order to compute the solution* $\mathbf{y}^{n+1}$ *of* (1.2)*,*

1) Solve the modified problem (1.3) *to find the defect step solution* $\tilde{\mathbf{y}}^{n+1}$*:*

$$\frac{\tilde{\mathbf{y}}^{n+1} - \tilde{\mathbf{y}}^n}{\Delta t} = \mathcal{F}(t_{n+1}, \tilde{\mathbf{y}}^n, \tilde{\mathbf{y}}^{n+1}).$$

2) Correct to get the structure of (1.2)*: the correction step solution* $\hat{\mathbf{y}}^{n+1}$ *satisfies*

$$\frac{\hat{\mathbf{y}}^{n+1} - \hat{\mathbf{y}}^n}{\Delta t} = \mathcal{F}(t_{n+1}, \hat{\mathbf{y}}^n, \hat{\mathbf{y}}^{n+1}) - \mathcal{F}(t_{n+1}, \tilde{\mathbf{y}}^n, \tilde{\mathbf{y}}^{n+1}) + F(t_{n+1}, \tilde{\mathbf{y}}^{n+1}).$$

3) This step is problem-specific: use the correction step solution $\hat{\mathbf{y}}^{n+1}$ *to linearize/decouple* (1.2)*. Solve to find the second correction solution* $\mathbf{y}^{n+1}$*. When decoupling, this step solves*

$$\frac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = \mathcal{F}(t_{n+1}, \hat{\mathbf{y}}^{n+1}, \mathbf{y}^{n+1}).$$

4) Update the correction step solution with the best available approximation: $\hat{\mathbf{y}}^{n+1} := \mathbf{y}^{n+1}$*.*

Note that the defect step is not being updated as it runs independently from the correction steps. This is in agreement with "mainstream" defect correction methods, [1], and it allows for efficient parallelization. At the same time, the second correction step solution $\mathbf{y}^{n+1}$ is used to update the correction step solution $\hat{\mathbf{y}}^{n+1}$ (hence, the name of the method), so the correction steps need to be run consecutively.

We will show with various numerical tests that the RC method does not suffer from asymptotic convergence. The qualitative and quantitative results throughout this paper are obtained at very large time step sizes (including one test with large and varying time steps), and they all show the superiority of the RC solution over the solution of the method used for linearization/decoupling of the original problem.

We see the immediate benefit of this paper in the application of RC to decoupling: in Section 4, a combination of Recursive Correction with the Refactorized Midpoint method (see the discussion below) is used to model MagnetoHydroDynamic flows. However, we also believe that the efficacy of the RC technique in linearizing the Navier-Stokes equations, Section 2, warrants further investigation.

The Refactorized Midpoint (RM) method, [5], stems from an observation that some second-order accurate methods (midpoint, trapezoidal) could be obtained by taking a half-step of the Forward Euler and half-step of the Backward Euler method, [8]. This possibility of building on the Backward Euler structure to produce a second-order accurate, symplectic method (midpoint rule) that is amenable to variable step size and is A- and B-stable, is especially important when the availability of time discretization techniques is limited by the complexity of the problem.

[1]In this paper, we present the results for the Backward Euler method, although we have also successfully tested the RC method for the linearization of the NSE with the Midpoint Rule for time discretization.

For a general first-order ODE $y'(t) = f(t, y(t))$, the Midpoint Rule seeks the solution at the new time level via

$$\frac{y^{n+1} - y^n}{\tau_n} = f(t_{n+1/2}, y^{n+1/2}), \qquad t_{n+1} = t_n + \tau_n, t_{n+1/2} = t_n + \frac{\tau_n}{2}.$$

It can be implemented by taking half-step of the Backward Euler method, followed by a half-step of the Forward Euler method:

$$\frac{y^{n+1/2} - y^n}{\tau_n/2} = f(t_{n+1/2}, y^{n+1/2}),$$
$$\frac{y^{n+1} - y^{n+1/2}}{\tau_n/2} = f(t_{n+1/2}, y^{n+1/2}).$$

This leads to the Refactorized Midpoint formulation

$$\begin{cases} \frac{y^{n+1/2} - y^n}{\tau_n/2} = f(t_{n+1/2}, y^{n+1/2}) \\ y^{n+1} = 2y^{n+1/2} - y^n \end{cases} \tag{1.4}$$

The Refactorized Midpoint method (1.4) was shown in [5] to be B-stable and second-order accurate. However, the drawback of this method is that the midpoint value $y^{n+1/2}$ used in the extrapolation step has to be obtained by a half-step of the Backward Euler method; if this is not computationally feasible and the Backward Euler method is replaced with another first-order accurate scheme - say, an implicit-explicit decoupling method - then the stability of the extrapolation step is reduced to that of a Forward Euler method. For this reason, Trenchea et al. proposed to iterate the solution of the decoupling method until it converges to the solution of the Backward Euler method, before it is followed up by the extrapolation step of (1.4). The authors used this approach, denoted by the Iterated Refactorized Midpoint, in MagnetoHydroDynamics [18] and in the Fluid-Structure Interaction problems [4]. In both problems, it required a substantial number of iterations per time step (from 3 iterations at small time steps to 9 iterations when the time step size is large). In Section 4, we will show that the Refactorized Midpoint with Recursive Correction (RMRC) method for the MHD is second-order accurate in time, and it requires only two iterations per time step (when parallelized), even when the time step seems prohibitively large.

2. Navier-Stokes: Linearization via Recursive Correction

We start with the semi-discrete (continuous in space, discretized in time) formulation of the Navier-Stokes equations, with the Backward Euler method for time discretization.[2] The method seeks the velocity-pressure pair (u^{n+1}, p^{n+1}) satisfying

$$\frac{u^{n+1} - u^n}{\Delta t} - \nu \Delta u^{n+1} + u^{n+1} \cdot \nabla u^{n+1} + \nabla p^{n+1} = f(t_{n+1}), \tag{2.1}$$
$$\nabla \cdot u^{n+1} = 0.$$

It is well established that this method is first-order accurate in time. Contrary to many existing defect correction approaches [1], Recursive Correction does not seek to improve the accuracy of (2.1). Instead, we propose that the RC method should be implemented when a solution of (2.1) needs to be recovered after some alteration: we will consider cases when a linearization is applied to (2.1) or when the Backward Euler method for a coupled system is ruined by a partitioning method.

[2] We have also implemented Recursive Correction for the Navier-Stokes equations with the Midpoint Rule for time discretization; the resulting convergence rates are very close to the ones we report here for the Backward Euler method, so we chose to not include them in this manuscript.

A linearization of (2.1) with the Newton's method computes several consecutive approximations per time step, until the stopping criterion is met. One step of the Newton's method seeks u^{n+1} via

$$\frac{u^{n+1}-u^n}{\Delta t} - \nu\Delta u^{n+1} + u^n\cdot\nabla u^{n+1} + u^{n+1}\cdot\nabla u^n - u^n\cdot\nabla u^n + \nabla p^{n+1} = f(t_{n+1}), \qquad \nabla\cdot u^{n+1} = 0. \tag{2.2}$$

We propose the following Recursive Correction method, which could be used in place of the Newton's method.

Algorithm 2.1. *In order to advance the solution u from time level t_n to t_{n+1}:*

1) Take one step of the Newton's method to compute the defect step solution $\tilde{u}^{n+1}$.

2) Compute the correction step solution $\hat{u}^{n+1}$, using $\hat{u}^n$ and $\tilde{u}^{n+1}$.

3) Compute the second correction u^{n+1} by linearizing the problem using the correction step solution: replace $u^{n+1}\cdot\nabla u^{n+1}$ with $\hat{u}^{n+1}\cdot\nabla u^{n+1}$.

4) Update the correction step solution: $\hat{u}^{n+1} := u^{n+1}$.

Remark 2.1. *Do NOT update the defect step solution: it runs independently of the correction steps and allows for parallelization, which effectively reduces the cost of the method to two steps of the Newton method per time step.*

The defect step solution is sought via

$$\frac{\tilde{u}^{n+1}-\tilde{u}^n}{\Delta t} - \nu\Delta \tilde{u}^{n+1} + \tilde{u}^n\cdot\nabla \tilde{u}^{n+1} + \tilde{u}^{n+1}\cdot\nabla \tilde{u}^n - \tilde{u}^n\cdot\nabla \tilde{u}^n + \nabla \tilde{p}^{n+1} = f(t_{n+1}), \qquad \nabla\cdot \tilde{u}^{n+1} = 0. \tag{2.3}$$

The correction step seeks to restore the structure of (2.1) while keeping the left-hand side of (2.3). It computes $\hat{u}^{n+1}$ satisfying

$$\begin{aligned} \frac{\hat{u}^{n+1}-\hat{u}^n}{\Delta t} - \nu\Delta \hat{u}^{n+1} + \hat{u}^n\cdot\nabla \hat{u}^{n+1} + \hat{u}^{n+1}\cdot\nabla \hat{u}^n - \hat{u}^n\cdot\nabla \hat{u}^n + \nabla \hat{p}^{n+1} &= f(t_{n+1}) \\ +\tilde{u}^n\cdot\nabla \tilde{u}^{n+1} + \tilde{u}^{n+1}\cdot\nabla \tilde{u}^n - \tilde{u}^n\cdot\nabla \tilde{u}^n - \tilde{u}^{n+1}\cdot\nabla \tilde{u}^{n+1}&, \\ \nabla\cdot \hat{u}^{n+1} &= 0. \end{aligned} \tag{2.4}$$

If both $\hat{u}$ and $\tilde{u}$ are replaced with u, (2.4) becomes (2.1). The four nonlinear terms in the right hand side of (2.4) could be combined into $-(\tilde{u}^{n+1}-\tilde{u}^n)\cdot\nabla(\tilde{u}^{n+1}-\tilde{u}^n)$.

Finally, the second correction linearizes (2.1) using the solution of (2.4):

$$\frac{u^{n+1}-u^n}{\Delta t} - \nu\Delta u^{n+1} + \hat{u}^{n+1}\cdot\nabla u^{n+1} + \nabla p^{n+1} = f(t_{n+1}), \qquad \nabla\cdot u^{n+1} = 0. \tag{2.5}$$

The RC method (2.3)-(2.5) is designed to run in parallel on three cores. The computational cost includes the storage of the defect solution (because the first "defect" core will be done in half the time needed for the two other cores); the solution will be passed from core 1 to core 2 to core 3, and back to core 2; the CPU time is roughly equal to two steps of the Newton's method per time step (plus the time needed for the data exchange between the cores).

2.1. Numerical Test: Linearization

An extensive investigation of the RC capabilities in linearization would be needed to find a place for the RC within the existing variety of linearization techniques. Herein, we do not test the RC on problems where the Newton's method fails; we do not compare the RC to, e.g., Andersen acceleration; instead, we aim to show the quadratic convergence of the RC linearization on one benchmark problem.

Consider the 2-D traveling wave test problem (see, e.g., [1, 17]) in $\Omega = [0.5, 1.5] \times [0.5, 1.5]$ at Reynolds number $Re = 1,000$ with the known true solution

$$\begin{aligned} u_1 &= \tfrac{3}{4} + \tfrac{1}{4}\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2 t\nu}, \\ u_2 &= \tfrac{3}{4} - \tfrac{1}{4}\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2 t\nu}, \\ p &= \tfrac{-1}{64}(\cos(4\pi(x-t)) + \cos(4\pi(y-t)))e^{-16\pi^2 t\nu}. \end{aligned}$$

Set the problem data (initial and boundary conditions, forcing) to match the true solution and run the RC linearization (2.3)-(2.5). On a fixed mesh with diameter $h = 1/64$, we run the method at various values of the time step. The defect and corrected solutions[3] are compared to the solution obtained by the Newton's method with tolerance 10^{-12}. Tables 1-2 show the errors in the $L^2 - L^2$ and $L^2 - H^1$ norms, along with the convergence rates. The quadratic convergence established below refers to $\alpha = 2$ in $\|\text{true solution} - \text{defect solution}\|^\alpha = \|\text{true solution} - \text{corrected solution}\|$. Note the somewhat unconventional way of using the table data to compute convergence rates: we use the data in a given row (rather than two neighboring rows in a given column) to obtain the RC convergence at the given fixed value of the time step.

Δt	$\|\|u_{Newton} - \tilde{u}\|\|_{L^2(0,T;L^2(\Omega))}$	$\|\|u_{Newton} - u\|\|_{L^2(0,T;L^2(\Omega))}$	Rate
8h	0.00678955	1.11E-05	2.29
4h	0.00141613	7.28E-07	2.15
2h	0.000251598	3.18E-08	2.08
h	4.13E-05	1.08E-09	2.05
h/2	6.37E-06	3.29E-11	2.02

Table 1: RC Errors (defect and second correction) in the $L^2(0,T;L^2(\Omega))$-norm, $h = 1/64$, $T = 1$, $\nu = 0.001$.

Δt	$\|\|u_{Newton} - \tilde{u}\|\|_{L^2(0,T;L^2(\Omega))}$	$\|\|u_{Newton} - u\|\|_{L^2(0,T;L^2(\Omega))}$	Rate
8h	0.395626	0.000467105	8.27
4h	0.0862457	3.01E-05	4.25
2h	0.0147464	1.58E-06	3.17
h	0.00225915	7.76E-08	2.69
h/2	0.000329449	3.34E-09	2.43

Table 2: RC Errors (defect and second correction) in the $L^2(0,T;H^1(\Omega))$-norm, $h = 1/64$, $T = 1$, $\nu = 0.001$.

Note that the main flaw of defect correction is not present in the RC: the expected convergence rates are immediately achieved at the large values of the time step, and there is no "asymptotic convergence".

3. Decoupling with Recursive Correction - MagnetoHydroDynamics

We move to a situation even more fitting for the RC approach - the partitioning of coupled CFD systems. We will focus on the MagnetoHydroDynamic flows. The quantities of interest are the velocity field u and the magnetic field B (or the fluctuations b around the mean magnetic field B_0), but we will consider the MHD equations written in the Elsässer variables $z_+ = u + b, z_- = u - b$.

The motion of an electrically conducting incompressible fluid flow in the presence of a magnetic field is governed by the following equations for the velocity field u, magnetic field B, and pressure p:

[3] "Corrected solution" refers to the second correction (2.5).

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - (B\cdot\nabla)B - \nu\Delta u + \nabla p = f, \qquad \nabla\cdot u = 0,$$
$$\frac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u - \nu_m\Delta B + \nabla q = g, \qquad \nabla\cdot B = 0,$$

Split the total magnetic field into mean and fluctuations, $B = B_\circ + b$. Adding and subtracting the MHD momentum equations leads to the formulation in the Elsässer variables:

$$\frac{\partial z_\pm}{\partial t} \mp (B_\circ\cdot\nabla) z_\pm + (z_\mp\cdot\nabla) z_\pm - \frac{\nu+\nu_m}{2}\Delta z_\pm - \frac{\nu-\nu_m}{2}\Delta z_\mp + \nabla p_\pm = f_\pm, \tag{3.1}$$

along with $\nabla\cdot z_\pm = 0$.

The semi-discrete (continuous in space) formulation with the Backward Euler method for time discretization seeks $z_\pm^{n+1}$ satisfying

$$\begin{aligned}\frac{z_+^{n+1}-z_+^n}{\Delta t} - B_0\cdot\nabla z_+^{n+1} + z_-^{n+1}\cdot\nabla z_+^{n+1} - \frac{\nu+\nu_m}{2}\Delta z_+^{n+1}& \\ -\frac{\nu-\nu_m}{2}\Delta z_-^{n+1} + \nabla p_+^{n+1} &= f_+(t_{n+1}), \\ \nabla\cdot z_+^{n+1} &= 0, \\ \frac{z_-^{n+1}-z_-^n}{\Delta t} + B_0\cdot\nabla z_-^{n+1} + z_+^{n+1}\cdot\nabla z_-^{n+1} - \frac{\nu+\nu_m}{2}\Delta z_-^{n+1}& \\ -\frac{\nu-\nu_m}{2}\Delta z_+^{n+1} + \nabla p_-^{n+1} &= f_-(t_{n+1}), \\ \nabla\cdot z_-^{n+1} &= 0.\end{aligned} \tag{3.2}$$

The implicit-explicit (IMEX) decoupled model, introduced and proven to be unconditionally stable and first-order accurate in time in [6], reads:

$$\begin{aligned}\frac{z_+^{n+1}-z_+^n}{\Delta t} - B_0\cdot\nabla z_+^{n+1} + z_-^{n}\cdot\nabla z_+^{n+1} - \frac{\nu+\nu_m}{2}\Delta z_+^{n+1}& \\ -\frac{\nu-\nu_m}{2}\Delta z_-^{n} + \nabla p_+^{n+1} &= f_+(t_{n+1}), \\ \nabla\cdot z_+^{n+1} &= 0, \\ \frac{z_-^{n+1}-z_-^n}{\Delta t} + B_0\cdot\nabla z_-^{n+1} + z_+^{n}\cdot\nabla z_-^{n+1} - \frac{\nu+\nu_m}{2}\Delta z_-^{n+1}& \\ -\frac{\nu-\nu_m}{2}\Delta z_+^{n} + \nabla p_-^{n+1} &= f_-(t_{n+1}), \\ \nabla\cdot z_-^{n+1} &= 0.\end{aligned} \tag{3.3}$$

We proceed with the formulation of the general RC Algorithm 1.1 that is tailored to the IMEX decoupling method (3.3).

Algorithm 3.1. *In order to advance the solution $z_\pm$ from time level t_n to t_{n+1}:*

1) Take one step of the IMEX decoupling method to compute the defect step solution $\tilde{z}_\pm^{n+1}$.

2) Compute the correction step solution $\hat{z}_\pm^{n+1}$, using $\hat{z}_\pm^n$ and $\tilde{z}_\pm^{n+1}$. The correction aims to restore the coupled structure of the original system (3.2)*.*

3) Compute the second correction $z_\pm^{n+1}$ by decoupling (3.2) *using the correction step solution: in the IMEX decoupling method, replace all entries at previous time levels with $\hat{z}_\pm^{n+1}$.*

4) Update the correction step solution: $\hat{z}_\pm^{n+1} := z_\pm^{n+1}$.

The RC approach does not seek to improve the temporal accuracy of (3.2). Rather, the Recursive Correction aims to produce a solution that would be closer to the solution of (3.2) (albeit, still a first-order accurate approximation to the true solution of the continuous MHD system). To that end, we compute three approximations per time step: the defect step solution $\tilde{z}_\pm$, followed by the correction step solution $\hat{z}_\pm$ and, finally, the second correction solution $z_\pm$. The cost, reduced by parallelization, is comparable to two runs of the decoupling method per time step.

The defect step solution is the solution of (3.3):

$$
\begin{aligned}
\frac{\tilde{z}_+^{n+1} - \tilde{z}_+^n}{\Delta t} - B_0 \cdot \nabla \tilde{z}_+^{n+1} + \tilde{z}_-^n \cdot \nabla \tilde{z}_+^{n+1} - \frac{\nu + \nu_m}{2} \Delta \tilde{z}_+^{n+1} \\
- \frac{\nu - \nu_m}{2} \Delta \tilde{z}_-^n + \nabla \tilde{p}_+^{n+1} = f_+(t_{n+1}), \\
\nabla \cdot \tilde{z}_+^{n+1} = 0, \\
\frac{\tilde{z}_-^{n+1} - \tilde{z}_-^n}{\Delta t} + B_0 \cdot \nabla \tilde{z}_-^{n+1} + \tilde{z}_+^n \cdot \nabla \tilde{z}_-^{n+1} - \frac{\nu + \nu_m}{2} \Delta \tilde{z}_-^{n+1} \\
- \frac{\nu - \nu_m}{2} \Delta \tilde{z}_+^n + \nabla \tilde{z}_-^{n+1} = f_-(t_{n+1}), \\
\nabla \cdot \tilde{z}_-^{n+1} = 0.
\end{aligned}
\tag{3.4}
$$

The correction aims to restore the structure of (3.2), while keeping the left hand side of (3.4).

$$
\begin{aligned}
\frac{\hat{z}_+^{n+1} - \hat{z}_+^n}{\Delta t} - B_0 \cdot \nabla \hat{z}_+^{n+1} + \hat{z}_-^n \cdot \nabla \hat{z}_+^{n+1} - \frac{\nu + \nu_m}{2} \Delta \hat{z}_+^{n+1} \\
- \frac{\nu - \nu_m}{2} \Delta \hat{z}_-^n + \nabla \hat{p}_+^{n+1} = f_+(t_{n+1}) + (\tilde{z}_-^n - \tilde{z}_-^{n+1}) \cdot \nabla \tilde{z}_+^{n+1}, \\
\nabla \cdot \hat{z}_+^{n+1} = 0, \\
\frac{\hat{z}_-^{n+1} - \hat{z}_-^n}{\Delta t} + B_0 \cdot \nabla \hat{z}_-^{n+1} + \hat{z}_+^n \cdot \nabla \hat{z}_-^{n+1} - \frac{\nu + \nu_m}{2} \Delta \hat{z}_-^{n+1} \\
- \frac{\nu - \nu_m}{2} \Delta \hat{z}_+^n + \nabla \hat{z}_-^{n+1} = f_-(t_{n+1}) + (\tilde{z}_+^n - \tilde{z}_+^{n+1}) \cdot \nabla \tilde{z}_-^{n+1}, \\
\nabla \cdot \hat{z}_-^{n+1} = 0.
\end{aligned}
\tag{3.5}
$$

Finally, the second correction linearizes/decouples (3.2) using the corrected solution of (3.5).

$$
\begin{aligned}
\frac{z_+^{n+1} - z_+^n}{\Delta t} - B_0 \cdot \nabla z_+^{n+1} + \hat{z}_-^{n+1} \cdot \nabla z_+^{n+1} - \frac{\nu + \nu_m}{2} \Delta z_+^{n+1} \\
- \frac{\nu - \nu_m}{2} \Delta \hat{z}_-^{n+1} + \nabla p_+^{n+1} = f_+(t_{n+1}), \\
\nabla \cdot z_+^{n+1} = 0, \\
\frac{z_-^{n+1} - z_-^n}{\Delta t} + B_0 \cdot \nabla z_-^{n+1} + \hat{z}_+^{n+1} \cdot \nabla z_-^{n+1} - \frac{\nu + \nu_m}{2} \Delta z_-^{n+1} \\
- \frac{\nu - \nu_m}{2} \Delta \hat{z}_+^{n+1} + \nabla p_-^{n+1} = f_-(t_{n+1}), \\
\nabla \cdot z_-^{n+1} = 0.
\end{aligned}
\tag{3.6}
$$

The correction steps (3.5) use the best available approximation $z_\pm^n$ from the previous time level: $\hat{z}_\pm^n = z_\pm^n$. The defect steps run independently: $\tilde{z}_+^n$ is used to find $\tilde{z}_+^{n+1}$.

3.1. RC for MHD - Numerical Test

The effects of the RC method will become even more clear in the next section, where we add the Refactorized Midpoint method, building on the accurate approximation of the coupled Backward Euler solution by the RC method. However, even the standalone RC method will be shown to outperform the MHD IMEX decoupling method (3.3) when the time step size is large.

Consider the MHD flow past a step (see, e.g., [1, 2, 3, 20]). The domain is $\Omega = [0, 40] \times [0, 10]$ with a step at $[5, 6] \times [0, 1]$. The flow at $B_0 =< 0, 0 >, Re = 600, Re_m = 700$ enters the domain from the left; as it moves past the step, vortices start to form behind the step. These vortices then grow in size, detach, and travel to the right. We resolve the flow using a very small time step size; the obtained snapshot of the velocity field at time $T = 40$ is shown in Figure 1, and we regard this as the true solution.

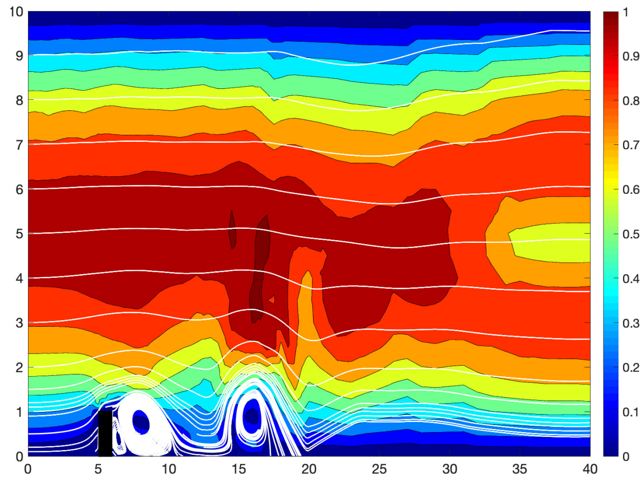
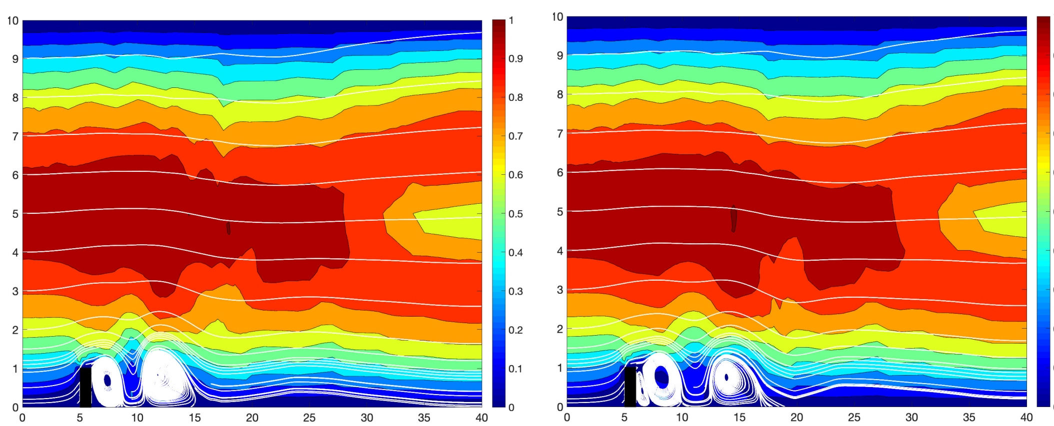


Figure 1: "True" solution, computed via the IMEX scheme with $\Delta t = 0.005$, at $T = 40$.

Next, we apply the RC method with a very coarse time step, $\Delta t = 0.5$. The IMEX solution (defect step) exhibits the qualitatively incorrect behavior, with only two vortices formed (instead of three), and the traveling vortex not traveling far enough. The RC solution (second correction step) is much more accurate, even though the traveling vortex does not quite reach the position suggested by the true solution, see Figure 2.



(a) Defect Step Solution (IMEX) at $T = 40$

(b) Corrected Solution (RC) at $T = 40$

Figure 2: Defect Step solution (left) and Correction Step solution (right) at $T = 40$. The time step is $\Delta t = 0.5$.

4. Refactorized Midpoint via Recursive Correction: IMEX → Correct to BE → RM

The following Iterated Refactorized Midpoint method was proposed for the MHD system by Trenchea, [18]: Iterate

$$\begin{aligned} \frac{z_+^{(k)} - z_+^n}{\Delta t/2} - B_0 \cdot \nabla z_+^{(k)} + z_-^{(k-1)} \cdot \nabla z_+^{(k)} - \frac{\nu + \nu_m}{2} \Delta z_+^{(k)} & \\ - \frac{\nu - \nu_m}{2} \Delta z_-^{(k-1)} + \nabla p_+^{(k)} &= f_+(t_{n+1/2}), \\ \nabla \cdot z_+^{(k)} &= 0, \\ \frac{z_-^{(k)} - z_-^n}{\Delta t/2} + B_0 \cdot \nabla z_-^{(k)} + z_+^{(k-1)} \cdot \nabla z_-^{(k)} - \frac{\nu + \nu_m}{2} \Delta z_-^{(k)} & \\ - \frac{\nu - \nu_m}{2} \Delta z_+^{(k-1)} + \nabla p_-^{(k)} &= f_-(t_{n+1/2}), \\ \nabla \cdot z_-^{(k)} &= 0 \end{aligned} \tag{4.1}$$

until convergence to the solution of the Backward Euler method for the coupled MHD at half-step. Then extrapolate $z_\pm^{n+1} = 2z_\pm^{n+1/2} - z_\pm^n$ to find the solution at the next time level.

The authors of [18] (and [4], where this Iterated Refactorized Midpoint method is used to model Fluid-Structure Interaction problems) aimed to decouple the problem using an unconditionally stable method (IMEX method (3.3), in the case of MHD modeling), but enjoy all the benefits of the Midpoint Rule - a symplectic second-order accurate method, that is A- and B-stable and allows for variable size time stepping. However, the price to pay was steep: three iterations of (4.1) per time step were needed if the time step size was sufficiently small; for larger time steps, up to nine iterations of (4.1) were required for convergence.

We will show in this section that, starting from the IMEX partitioning method (3.3), the Recursive Correction gets us sufficiently close to the solution of the Backward Euler method for the coupled MHD system, so that the extrapolation step of the Refactorized Midpoint method is stable even for very large time step size. The partial parallelization of the RC reduces the CPU time needed by this new method, which we call Refactorized Midpoint with Recursive Correction (RMRC), to approximately two iterations of (3.3) per time step. Thus, we get the second-order accuracy, nonlinear stability, and the symplectic nature of the Midpoint method, for the price of running the decoupling method twice per time step - even when the time step is prohibitively large for the decoupling method itself.

The RMRC method seeks the solution at the next time level t_{n+1} via the following procedure.

Algorithm 4.1. *1) Take a half-step with the chosen first-order accurate decoupling method to obtain the (defect) solution at time $t_{n+1/2}$, $\tilde{u}^{n+1/2}$.*
2) Use defect correction to produce a better approximation of the solution at $t_{n+1/2}$, $\hat{u}^{n+1/2}$. Note that the correction depends on the quantity of interest; herein we do not seek to improve the accuracy of the defect step solution. Instead, we correct to the (unknown) solution of the fully coupled Backward Euler method.
3) Run the decoupling method of step (1) again, but use the corrected solution $\hat{u}^{n+1/2}$ in place of the terms that were evaluated at the previous time level t_n. This creates the decoupled Backward Euler method for finding the (second correction) solution at $t_{n+1/2}$, $u^{n+1/2}$.
4) Extrapolate $u^{n+1} = 2u^{n+1/2} - u^n$, where $u^{n+1/2}$ is the solution from step (3). Set the correction solution from step (2) at the next time level equal to the solution of step (3): $\hat{u}^{n+1} := u^{n+1}$.
5) Run the decoupling method of step (1) again to obtain the defect solution at the new time level t_{n+1}. The seemingly better choice of using the extrapolated solution from the previous time level as the starting value for the half-step of the defect method renders the RMRC unstable.

The partial parallelization of RMRC, mentioned above, refers to running the method on three cores in the following manner. While the defect solution $\tilde{u}^{n+1}$ is computed at time t_{n+1} on core 1, the correction step runs on core 2 to compute $\hat{u}^{n+1/2}$ at time $t_{n+1/2}$. Next, while core 1 is computing the defect solution $\tilde{u}^{n+3/2}$ at time $t_{n+3/2}$, core 3 computes the second correction solution $u^{n+1/2}$ and extrapolates $u^{n+1} = 2u^{n+1/2} - u^n$. This amounts to roughly the CPU time needed for two runs of the defect step per time level (plus the time to send the computed solution from one core to another), because all three steps (defect and two corrections) require similar CPU time.

We also wish to point out that the proposed RMRC method aims to improve the results of the RC decoupling method of Section 3, at no extra computational cost: the increased accuracy (and other benefits of the symplectic Midpoint method) is achieved by the extrapolation $u^{n+1} = 2u^{n+1/2} - u^n$, which is the only difference between the RC and RMRC.

Applying the RMRC method to the MHD system results in the defect step

$$\begin{aligned}\frac{\tilde{z}_+^{n+1/2} - \tilde{z}_+^n}{\Delta t/2} - B_0 \cdot \nabla \tilde{z}_+^{n+1/2} + \tilde{z}_-^n \cdot \nabla \tilde{z}_+^{n+1/2} - \frac{\nu + \nu_m}{2} \Delta \tilde{z}_+^{n+1/2} \\ - \frac{\nu - \nu_m}{2} \Delta \tilde{z}_-^n + \nabla \tilde{p}_+^{n+1/2} = f_+(t_{n+1/2}), \\ \nabla \cdot \tilde{z}_+^{n+1/2} = 0, \\ \frac{\tilde{z}_-^{n+1/2} - \tilde{z}_-^n}{\Delta t/2} + B_0 \cdot \nabla \tilde{z}_-^{n+1/2} + \tilde{z}_+^n \cdot \nabla \tilde{z}_-^{n+1/2} - \frac{\nu + \nu_m}{2} \Delta \tilde{z}_-^{n+1/2} \\ - \frac{\nu - \nu_m}{2} \Delta \tilde{z}_+^n + \nabla \tilde{p}_-^{n+1/2} = f_-(t_{n+1/2}), \\ \nabla \cdot \tilde{z}_-^{n+1/2} = 0,\end{aligned} \tag{4.2}$$

followed by the correction at midstep

$$\begin{aligned}\frac{\hat{z}_+^{n+1/2} - \hat{z}_+^n}{\Delta t/2} - B_0 \cdot \nabla \hat{z}_+^{n+1/2} + \hat{z}_-^n \cdot \nabla \hat{z}_+^{n+1/2} - \frac{\nu + \nu_m}{2} \Delta \hat{z}_+^{n+1/2} \\ - \frac{\nu - \nu_m}{2} \Delta \hat{z}_-^n + \nabla \hat{p}_+^{n+1/2} = f_+(t_{n+1/2}) \\ + \left(\tilde{z}_-^n - \tilde{z}_-^{n+1/2}\right) \cdot \nabla \tilde{z}_+^{n+1/2} - \frac{(\nu - \nu_m)}{2} \Delta \left(\tilde{z}_-^n - \tilde{z}_-^{n+1/2}\right), \\ \nabla \cdot \hat{z}_+^{n+1/2} = 0, \\ \frac{\hat{z}_-^{n+1/2} - \hat{z}_-^n}{\Delta t/2} + B_0 \cdot \nabla \hat{z}_-^{n+1/2} + \hat{z}_+^n \cdot \nabla \hat{z}_-^{n+1/2} - \frac{\nu + \nu_m}{2} \Delta \hat{z}_-^{n+1/2} \\ - \frac{\nu - \nu_m}{2} \Delta \hat{z}_+^n + \nabla \hat{p}_-^{n+1/2} = f_-(t_{n+1/2}) \\ + \left(\tilde{z}_+^n - \tilde{z}_+^{n+1/2}\right) \cdot \nabla \tilde{z}_-^{n+1/2} - \frac{(\nu - \nu_m)}{2} \Delta \left(\tilde{z}_+^n - \tilde{z}_+^{n+1/2}\right), \\ \nabla \cdot \hat{z}_-^{n+1/2} = 0.\end{aligned} \tag{4.3}$$

Now we feed the corrected mid-step solution $\hat{z}_\pm^{n+1/2}$ to the IMEX method, resulting in the decoupled Backward Euler method. The solution $z_\pm^{n+1/2}$ is sought via

$$
\begin{aligned}
\frac{z_+^{n+1/2}-z_+^n}{\Delta t/2} - B_0\cdot\nabla z_+^{n+1/2} + \hat{z}_-^{n+1/2}\cdot\nabla z_+^{n+1/2} - \frac{\nu+\nu_m}{2}\Delta z_+^{n+1/2}& \\
-\frac{\nu-\nu_m}{2}\Delta \hat{z}_-^{n+1/2} + \nabla p_+^{n+1/2} &= f_+(t_{n+1/2}),\\
\nabla\cdot z_+^{n+1/2} &= 0,\\
\frac{z_-^{n+1/2}-z_-^n}{\Delta t/2} + B_0\cdot\nabla z_-^{n+1/2} + \hat{z}_+^{n+1/2}\cdot\nabla z_-^{n+1/2} - \frac{\nu+\nu_m}{2}\Delta z_-^{n+1/2}& \\
-\frac{\nu-\nu_m}{2}\Delta \hat{z}_+^{n+1/2} + \nabla p_-^{n+1/2} &= f_-(t_{n+1/2}),\\
\nabla\cdot z_-^{n+1/2} &= 0.
\end{aligned}
\tag{4.4}
$$

The solution at time level t_{n+1} is obtained via $z_\pm^{n+1} = 2z_\pm^{n+1/2} - z_\pm^n$. The corrected solution $\hat{z}_\pm^{n+1}$ is set to be equal to the best available data, $\hat{z}_\pm^{n+1} = z_\pm^{n+1}$. The defect solution at time t_{n+1} is computed by another (half-step) run of (4.2).

Remark 4.1. *We have investigated three approaches to updating the solutions at the new time level. The first approach, where both the defect and the correction step solutions at the new level are set equal to the second correction solution, $\tilde{z}_\pm^{n+1} = \hat{z}_\pm^{n+1} = z_\pm^{n+1}$, turned out to become unstable after several refinements of the time step. The second approach, without any updates, computed the defect solution $\tilde{z}_\pm^{n+1}$ and the corrected solution $\hat{z}_\pm^{n+1}$ by taking another half-step of* (4.2) *and* (4.3)*, respectively. This approach converged to the true solution, but at a slower rate. Finally, the approach where the correction step solution at the new level is set equal to the second correction solution, but the defect step approximation runs separately with two iterations per time step, turned out to be the best of all three in terms of both stability and accuracy.*

4.1. RMRC for MHD: Numerical Tests

4.1.1. Convergence Rates

We consider an electrically conducting flow inducing a magnetic field, with the following true solution, [20]:

$$
\begin{aligned}
z_+ &= \begin{pmatrix} \frac{3}{4}+\frac{1}{4}\ \cos(2\pi(\mathrm{x}-\mathrm{t}))\sin(2\pi(y-t))e^{(-8\pi^2 t\nu)} + (1+y)^2 e^{t\nu_m} \\ -\frac{1}{4}\ \sin(2\pi(\mathrm{x}-\mathrm{t}))\cos(2\pi(y-t))e^{(-8\pi^2 t\nu)} + (1+x)^2 e^{t\nu_m} \end{pmatrix}\\
z_- &= \begin{pmatrix} \frac{3}{4}+\frac{1}{4}\ \cos(2\pi(\mathrm{x}-\mathrm{t}))\sin(2\pi(y-t))e^{(-8\pi^2 t\nu)} - (1+y)^2 e^{t\nu_m} \\ -\frac{1}{4}\ \sin(2\pi(\mathrm{x}-\mathrm{t}))\cos(2\pi(y-t))e^{(-8\pi^2 t\nu)} - (1+x)^2 e^{t\nu_m} \end{pmatrix}\\
p_+ &= p_- = -\frac{1}{64}\ \cos(4\pi(\mathrm{x}-\mathrm{t}))\cos(4\pi(y-t))e^{(-16\pi^2 t\nu)}.
\end{aligned}
$$

N	$\|\|Z^+-\tilde{z}^+\|\|$	$\|\|Z^+-z^+\|\|$	$\|\|Z^--\tilde{z}^-\|\|$	$\|\|Z^--z^-\|\|$
8	0.0418466	0.0315979	0.0414538	0.0319473
16	0.0262091	0.00778904	0.0260164	0.00788757
32	0.0151651	0.00200411	0.0150771	0.00203021
64	0.00827474	0.000493646	0.00823708	0.000499663
128	0.00434276	0.000114429	0.00432647	0.000115728

Table 3: $L^2(0,T;L^2(\Omega))$-norm of the RMRC Errors. $\Delta t = h = 1/N$, $T = 1$, $\nu = 0.1$ and $\nu_m = 1$.

N	$\|\|Z^+ - \tilde{z}^+\|\|$	$\|\|Z^+ - z^+\|\|$	$\|\|Z^- - \tilde{z}^-\|\|$	$\|\|Z^- - z^-\|\|$
16	0.68	2.02	0.67	2.02
32	0.79	1.96	0.79	1.96
64	0.87	2.02	0.87	2.02
128	0.93	2.11	0.93	2.11

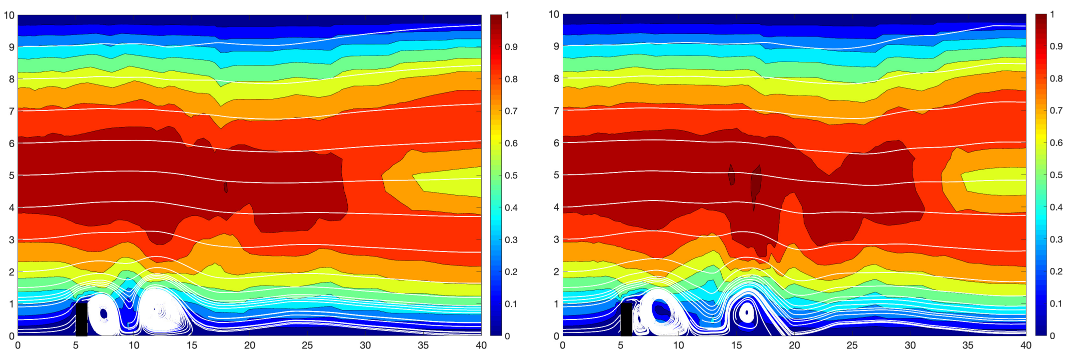
Table 4: Convergence Rates for Table 3.

N	$\|\|Z^+ - \tilde{z}^+\|\|$	$\|\|Z^+ - z^+\|\|$	$\|\|Z^- - \tilde{z}^-\|\|$	$\|\|Z^- - z^-\|\|$
8	0.0406775	0.0318598	0.052857	0.0420418
16	0.0212132	0.00637767	0.0317894	0.0100264
32	0.0108842	0.00152742	0.0174436	0.00243905
64	0.00554414	0.000380684	0.0091645	0.000595821
128	0.00279432	9.42034e-05	0.0047072	0.000146834

Table 5: $L^2(0, T; L^2(\Omega))$-norm of the RMRC Errors. $\Delta t = h = 1/N$, $T = 1$, $\nu = 10^{-4}$ and $\nu_m = 10^{-3}$.

4.1.2. Qualitative Test - Flow Past a Step

The velocity plots below are to be compared to the true solution in Figure 1.

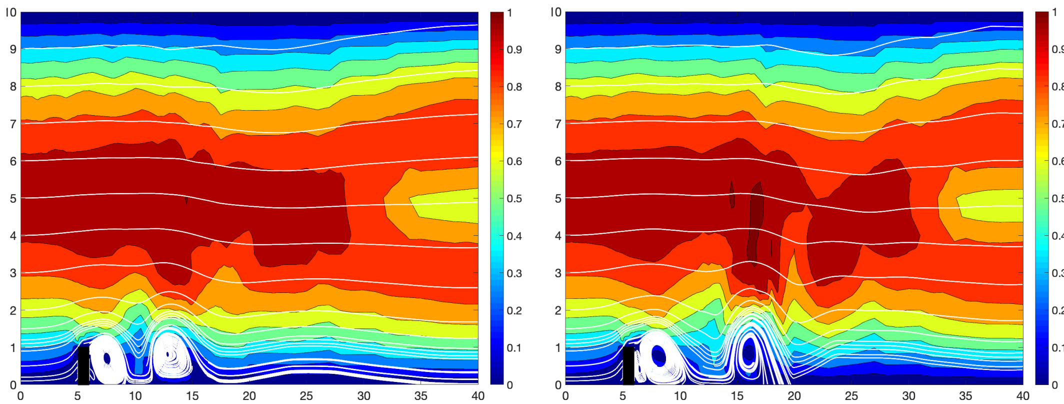


(a) Defect Step Solution (IMEX) at $T = 40$ (b) Corrected Solution (RC) at $T = 40$

Figure 3: Large time step $\Delta t = 1$. The IMEX solution (left) vs. the RMRC solution (right) at $T = 40$.

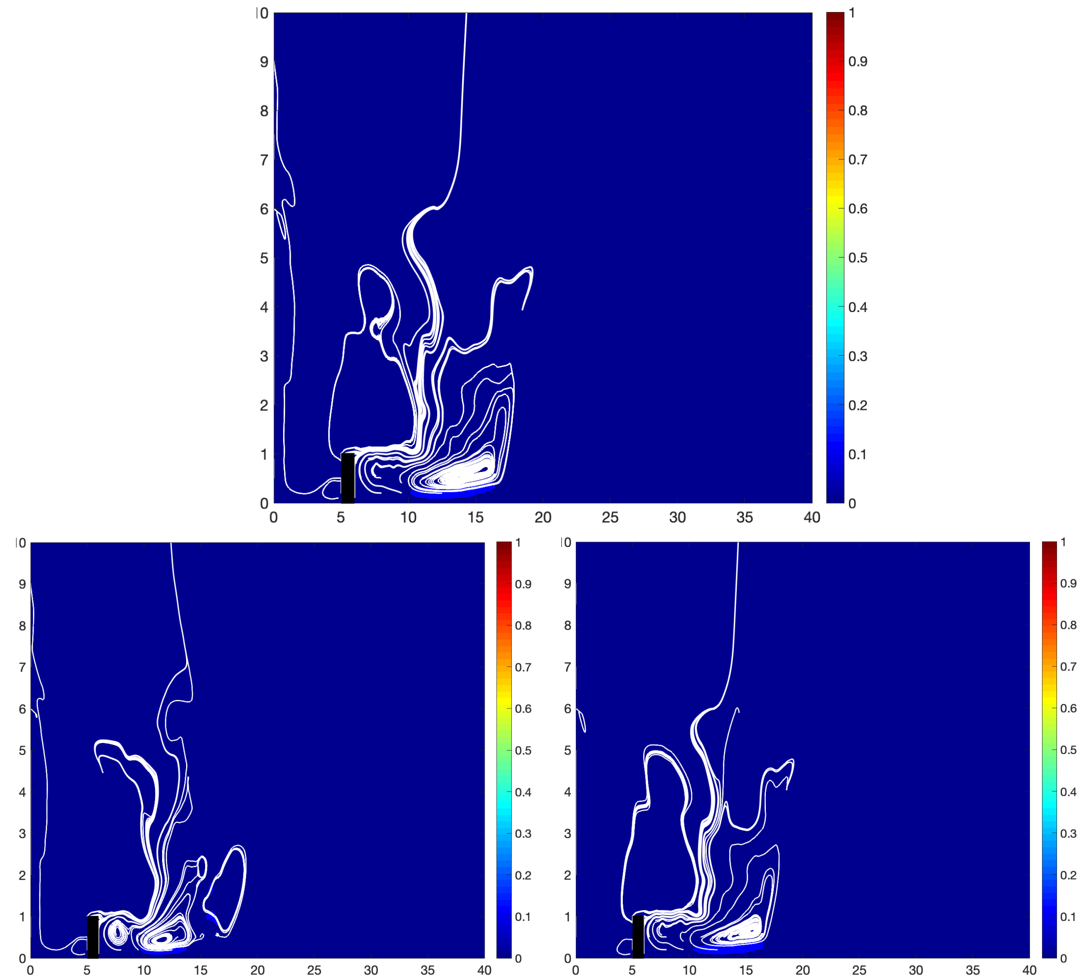
N	$\|\|Z^+ - \tilde{z}^+\|\|$	$\|\|Z^+ - z^+\|\|$	$\|\|Z^- - \tilde{z}^-\|\|$	$\|\|Z^- - z^-\|\|$
16	0.94	2.32	0.73	2.07
32	0.96	2.06	0.87	2.04
64	0.97	2.00	0.93	2.03
128	0.99	2.01	0.96	2.02

Table 6: Convergence Rates for Table 5.



(a) Defect Step Solution (IMEX) at $T = 40$ (b) Corrected Solution (RC) at $T = 40$

Figure 4: Variable time step $0.2 \leq \Delta t \leq 0.8$. The IMEX solution (left) vs. the RMRC solution (right) at $T = 40$.



(a) Defect Step Solution (IMEX), $B_0 =< -0.1, 0 >$ (b) Corrected Solution (RC), $B_0 =< -0.1, 0 >$

Figure 5: Magnetic Field plots at $T = 40$, $B_0 =< -0.1, 0 >$. True solution (top, $\Delta t = 0.005$) vs. the IMEX (left) and the RMRC solution (right) with $\Delta t = 0.5$.

5. Conclusions

We presented and investigated numerically a novel defect correction approach to linearization of the Navier-Stokes equations and the decoupling of the MagnetoHydroDynamic system. The approach, which we call Recursive Correction, differs from other defect correction techniques in that it does not seek to improve the accuracy of the defect step solution. Instead, it corrects to recover the structure of the original non-perturbed problem (nonlinear, coupled, etc.). The second correction then uses the corrected solution to linearize/decouple the original problem.

We showed this method to converge quadratically when linearizing the Navier-Stokes equations. We also applied it to the coupled Navier-Stokes system, MagnetoHydroDynamics, where this method improved the quality of the solution obtained by the existing implicit-explicit decoupling method. At no extra computational cost, we combined the Recursive Correction with the Refactorized Midpoint method to obtain a second-order accurate method that leads to the symplectic B-stable midpoint formulation of the coupled MHD, at the cost of running the decoupled implicit-explicit method twice per time step - a substantial improvement over the state-of-the-art Iterative Refactorized Midpoint method.

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