

# Analysis of a New Regularization Method for Ill-Conditioned Linear Systems

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## Introduction

Inverse problems consider observed effects and work backwards to solve for the experimental conditions, that is the causes, under a given mathematical model. The simplest inverse problems take the form of a linear problem  $Ax = b$  where  $A$  is a known matrix,  $b$  is the vector that contains measured data, and  $x$  is the vector that contains the unknown parameters; these parameters describe the causes, or experimental conditions, that are to be estimated.

When  $A$  is taken to be numerically singular or ill-conditioned, the linear system can be solved through the least squares method. The least squares method is very sensitive to any noise in the data which leads to a large contribution of error in the calculation of the parameters.<sup>2</sup> See figure 1 below.

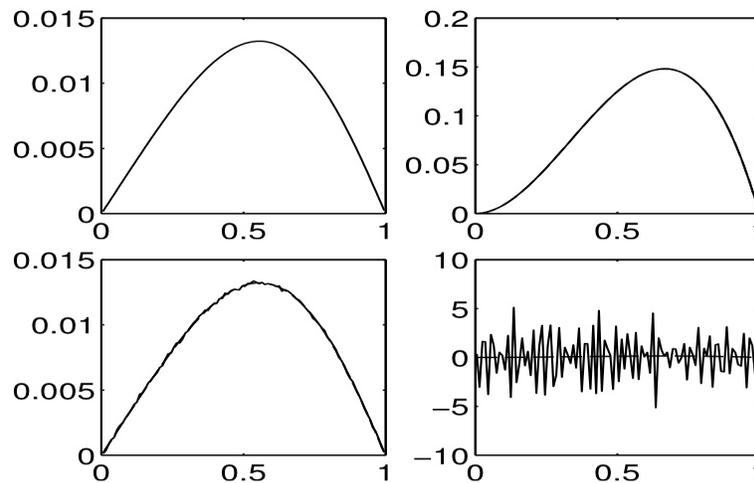


Figure 1: The top left graph displays an example of exact data with no noise contribution. The top right graph displays the exact solution to the linear inverse problem  $Ax = b$  corresponding to the data from the top left graph. Data with contribution from noise is graphed in the bottom left plot (notice that the noise is barely visible on this scale) and the solution corresponding to the noisy data is graphed in the bottom right plot. These figures illustrate the sensitivity to noise in data for linear inverse problems and the need for regularization.<sup>1</sup>

To correct for the error from the noise in the data, a regularization method is used, the most common of which is the Tikhonov Regularization Method (TRM). Rather than minimizing  $\|Ax - b\|^2$ , the TRM minimizes  $T(x) = \|Ax - b\|^2 + \lambda\|Bx\|^2$  which not only attempts to match the data but also seeks to reduce noise in the solution of  $x$ . The biggest difficulty with the TRM is choosing an appropriate regularization parameter,  $\lambda$ .<sup>3</sup>

Here we present a new regularization method, entitled the Multiplicative Regularization Method (MRM). The advantage of this new method is primarily that it eliminates the need for choosing a

regularization parameter by minimizing  $M(x) = \|Ax - b\|^2 \cdot \|Bx\|^2$ . Our interest for the summer is to establish a numerical and theoretical foundation for this new regularization method.

## Proposed Research

We first present our initial findings which illustrate a connection between the TRM and the MRM. We show for a specific regularization parameter  $\lambda$ , the TRM and the MRM are equivalent.

For minimization, we take the gradient of both functions and set them equal to zero. Note that  $\nabla\|Ax - b\|^2 = 2A^T Ax - 2A^T b$  and  $\nabla\|Bx\|^2 = 2B^T Bx$ . Using these formulas, we have

$$\begin{aligned}\nabla T(x) &= \nabla\|Ax - b\|^2 + \nabla\lambda\|Bx\|^2 \\ &= 2A^T Ax - 2A^T b + 2\lambda B^T Bx\end{aligned}$$

$$\begin{aligned}\nabla M(x) &= \nabla\|Ax - b\|^2 \cdot \|Bx\|^2 + \|Ax - b\|^2 \cdot \nabla\|Bx\|^2 \\ &= 2A^T Ax\|Bx\|^2 - 2A^T b\|Bx\|^2 + 2\|Ax - b\|^2 B^T Bx.\end{aligned}$$

Hence after algebraic manipulation we see the results for the Tikhonov method and the Multiplicative method.

$$\begin{aligned}\nabla T(x) = 0 &\Rightarrow A^T Ax + \lambda B^T Bx = A^T b \\ \nabla M(x) = 0 &\Rightarrow A^T Ax + \frac{\|Ax - b\|^2}{\|Bx\|^2} B^T Bx = A^T b.\end{aligned}$$

For any  $\lambda$ , we can calculate a corresponding solution  $x_\lambda$  using the TRM. When  $\lambda = \frac{\|Ax_\lambda - b\|^2}{\|Bx_\lambda\|^2}$ , then  $x_\lambda$  is also a solution for the MRM. This initial result is foundation for our investigation of the Multiplicative Regularization Method.

This project will address three questions:

1. For small values of noise, can we prove that the MRM has a unique solution of the form  $x = x_\lambda$ ,  $\lambda = \frac{\|Ax_\lambda - b\|^2}{\|Bx_\lambda\|^2}$ ?
2. Can we solve  $\lambda = \frac{\|Ax_\lambda - b\|^2}{\|Bx_\lambda\|^2}$  by fixed point iteration?
3. On standard test problems, how does this method compare numerically to other regularization methods? Can we analyze the effectiveness of the MRM theoretically?

## Outcomes

Studying the effectiveness of the Multiplicative Regularization Method will improve our understanding of the Tikhonov Regularization Method and guide us to developing a more refined regularization method. Developing an enhanced regularization method will benefit not only the field of mathematics but all areas of science that use regularization methods on inverse problems - particularly the medical field. Medical imaging uses regularization procedures to enhance brain scans and MRIs by eliminating white noise during the image processing, providing clearer images to prevent misdiagnoses.<sup>4</sup>

In addition to these contributions, this research will expand my undergraduate experience and prepare me for a future in graduate school. Research in any field, particularly mathematics, is important for broadening one's academic horizons and this research will enhance all of my experiences here at Michigan Technological University.

## References

- [1] Gockenbach, Mark S. *Finite-Dimensional Linear Algebra*. Boca Raton: Taylor and Francis Group, 2010.
- [2] Hansen, Per Christian. *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*. Philadelphia: Society for Industrial And Applied Mathematics, 1998.
- [3] Hofmann, Bernd. (2006) “Approximate Source Conditions in Tikhonov-Phillips Regularization and Consequences for Inverse Problems with Multiplication Operators.” *Mathematical Methods in the Applied Sciences*, 29: 1-4.
- [4] Zhang, Xin et. al. (2008) “Application of Tikhonov Regularization to Super-Resolution Reconstruction of Brain MRI Images.” *Springer Berlin/Heidelberg*, 4987: 51-56.