Sample questions

1.

Groups

(a) Isomorphism Theorems. Let $M$ and $N$ be normal subgroups of $G$ such that $G = MN$. Prove that $G/(M \cap N)$ is isomorphic to $(G/M) \times (G/N)$.

(b) Sylow theorem. Prove that if $|G| = 132$ then $G$ is not simple.

(c) Groups actions. Suppose $|G| = p^a$, where $p$ is a prime. Prove that every subgroup of index $p$ is normal in $G$.

(d) Cauchy-Frobenious-Burnside lemma. If there are $q$ colors available, prove that there are $(q^n + 2q^{(n^2 + 3)/4}) + q^{(n^2 + 1)/2}/4$ distinct $n \times n$ colored chessboards.

(e) Linear groups. Let $K$ be a field. Prove that $GL(n,K)$ is a semidirect product of $SL(n,K)$ by $K^\times = K - \{0\}$.

(f) Split extension. Construct a non-abelian group of order 75.

(g) Solvable groups. Suppose $|G| = pq$, where $p$ and $q$ are primes. Prove that $G$ is solvable.

(h) Finitely generated abelian groups. Let $G = \mathbb{Z}_{60} \times \mathbb{Z}_{45} \times \mathbb{Z}_{12}$ $\times$ $\mathbb{Z}_{36}$. Find the number of elements of order 2 in $G$.

2.

Rings.

(a) Polynomial rings. Let $f(x)$ be a polynomial in $F[x]$, where $F$ is a field. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

(b) Euclidean domains. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for any nonzero ideal $I$ of $\mathbb{Z}[i]$. ($\mathbb{Z}[i]$ is the ring of Gaussian integers).
Principal ideal domains. Let \( I = (2, 1 + \sqrt{-5}) \) be an ideal of \( \mathbb{Z}[\sqrt{-5}] \). Prove that \( I \) is not a principal ideal of \( \mathbb{Z}[\sqrt{-5}] \).

Unique factorization domains. Determine all the representations of the integer \( 2130797 = 17^2 \cdot 73 \cdot 101 \) as a sum of two squares.

3. Fields.

(a) Galois Theory and construction of Galois groups. Determine the Galois group of \( x^4 + 4x - 1 \).

(b) Finite fields. Write out the multiplication table for \( \mathbb{F}_4 \) (the field of 4 elements).

(c) Algebraic extensions. Prove that \( [Q(\sqrt{2} + \sqrt{3}) : Q] = 4 \), where \( Q \) is the field of rational numbers.