Qualifier Exam: Computation Theory

Computer Science

January 6, 2012

Do exactly 6 of the following 8 problems. You must do problem 8.

1. Let \( \Sigma \) be an alphabet and let \( w \) be a non-null string in \( \Sigma^* \). Define the language \( L_w \) recursively, as follows.
   \[
   w \in L_w, \\
   \text{if } s \in L_w, \text{ then } ss^R \in L_w,
   \]
   where \( s^R \) denotes the reversal of \( s \).
   Prove that for some choice of \( w \),
   i) \( L_w \) is regular, or
   ii) \( L_w \) is not regular and \( L_w \) is context-free, or
   iii) \( L_w \) is not context-free.

2. A useless state in a pushdown automaton is never entered on any input string. Show that the problem of determining whether a pushdown automaton has any useless states is decidable.

3. A queue automaton is identical to a pushdown automaton, except that the auxiliary data structure is a queue rather than a stack, that is, symbols added to the queue are removed in first-in, first-out order rather than last-in, first-out order.
   Consider the class of languages recognized by queue automata. How does it compare to other established classes of languages (e.g., regular, context-free, recursive, and recursively enumerable)? Justify your answer.

4. (a) Prove that it is undecidable whether a Turing machine \( M \), given input \( w \), enters all of its states.
    (b) Explain why Rice’s Theorem cannot be used to prove this result.

5. Prove that a language \( L \) is decidable iff \( L \) can be enumerated in lexicographic order.

6. Compare the classes of problems \( TIME(2^n) \) and \( \bigcup_k TIME(2^{n^k}) \). Are they equivalent? Justify your answer.

7. Recall that coNP is the set of languages \( L \) such that \( \overline{L} \in \text{NP} \).
   Prove that if \( \text{NP} \neq \text{coNP} \), then \( \text{P} \neq \text{NP} \).
8. A cubic graph is an undirected graph in which each vertex has degree three.

The CUBIC-VC problem is the VERTEX-COVER problem restricted to cubic graphs. Show that CUBIC-VC is NP-complete.

Hint: Give a reduction from VERTEX-COVER: Given a graph \((V, E)\) and a positive integer \(b\), is there a subset \(C\) of \(V\) of size \(b\) such that every edge in \(E\) has an endpoint in \(C\)?

Show how to "convert" an arbitrary graph \(G\) into a cubic graph, with an appropriate modification to the size bound \(b\). The problematic cases in \(G\) are vertices with degree 0, 1, 2 and \(> 3\).